CHAPTER 6

Hypothesis Testing

SAMPLING AND TESTS OF SIGNIFICANCE

6.1. POPULATION OR UNIVERSE

An aggregate of objects (animate or inanimate) under study is called **population or universe**. It is thus a collection of individuals or of their attributes (qualities) or of results of operations which can be numerically specified.

A universe containing a finite number of individuals or members is called a **finite inverse**. For example, the universe of the weights of students in a particular class or the universe of smokes in Rohtak district.

A universe with infinite number of members is known as an **infinite universe**. For example, the universe of pressures at various points in the atmosphere.

In some cases, we may be even ignorant whether or not a particular universe is infinite, *e.g.*, the universe of stars.

The universe of concrete objects is an **existent universe**. The collection of all possible ways in which a specified event can happen is called a **hypothetical universe**. The universe of heads and tails obtained by tossing a coin an infinite number of times (provided that it does not wear out) is a hypothetical one.

6.2. SAMPLING

The statistician is often confronted with the problem of discussing universe of which he cannot examine every member *i.e.*, of which complete enumeration is impracticable. For example, if we want to have an idea of the average per capita income of the people of India, enumeration of every earning individual in the country is a very difficult task. Naturally, the question arises: What can be said about a universe of which we can examine only a limited number of members? This question is the origin of the Theory of Sampling.

A finite sub-set of a universe is called a **sample.** A sample is thus a small portion of the universe. The number of individuals in a sample is called the **sample size**. The process of selecting a sample from a universe is called **sampling**.

The theory of sampling is a study of relationship existing between a population and samples drawn from the population. The fundamental object of sampling is to get as much information as possible of the whole universe by examining only a part of it. An attempt is thus made through sampling to give the maximum information about the parent universe with the minimum effort.

Sampling is quite often used in our day-to-day practical life. For example, in a shop we asses the quality of sugar, rice or any other commodity by taking only a handful of it from the bag and then decide whether to purchase it or not. A housewife normally tests the cooked products to find if they are properly cooked and contain the proper quantity of salt or sugar, by taking a spoonful of it.

6.3. PARAMETERS OF STATISTICS

The statistical constants of the population such as mean, the variance etc. are known as the parameters. The statistical concepts of the sample from the members of the sample to estimate the parameters of the population from which the sample has been drawn are known as *statistic*.

Population mean and variance are denoted by μ and σ^2, while those of the sample are given by $\bar{x}$, s^2.

6.4. STANDARD ERROR (S.E.)

The standard deviation of the sampling distribution of a statistic is known as the standard error (S.E.).

It plays an important role in the theory of large samples and it forms a basis of the testing of hypothesis. If t is any statistic, for large sample.

$$z = \frac{t - E(t)}{S.E(t)}$$ is normally distributed with mean 0 and variance unity.

For large sample, the standard errors of some of the well known statistic are listed below:

n—sample size; σ^2—population variance; s^2—sample variance; p—population proportion ; $Q = 1 - p$; n_1, n_2—are sizes of two independent random sample.

No.	*Statistic*	*Standard error*
1.	$\bar{x}$	$\sigma/\sqrt{n}$
2.	s	$\sqrt{\sigma^2/2n}$
3.	Difference of two sample means $\bar{x}_1 - \bar{x}_2$	$\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$
4.	Difference of two sample standard deviation $s_1 - s_2$	$\sqrt{\frac{\sigma_1^2}{2n_1} + \frac{\sigma_2^2}{2n_2}}$
5.	Difference of two sample proportions $p_1 - p_2$	$\sqrt{\frac{P_1Q_1}{n_1} + \frac{P_2Q_2}{n_2}}$
6.	Observed sample proportion p	$\sqrt{PQ/n}$

6.5. TEST OF SIGNIFICANCE

(P.T.U. 2008)

An important aspect of the sampling theory is to study the test of significance. Which will enable us to decide, on the basis of the results of the sample, whether

(*i*) *the deviation between the observed sample statistic and the hypothetical parameter value or*

(*ii*) *the deviation between two sample statistics is significant or might be attributed due to chance or the fluctuations of the sampling.*

For applying the tests of significance, we first set up a hypothesis which is a definite statement about the population parameter called **Null hypothesis** denoted by H_0.

Any hypothesis which is complementary to the null hypothesis (H_0) is called an **Alternative hypothesis** denoted by H_1.

For example if we want to test the null hypothesis that the population has a specified mean μ_0, then we have

$$H_0: \mu = \mu_0$$

Alternative hypothesis will be

(*i*) $H_1: \mu \neq \mu_0$ ($\mu > \mu_0$ *or* $\mu < \mu_0$) (two tailed alternative hypothesis).

(*ii*) $H_1: \mu > \mu_0$ (right tailed alternative hypothesis (*or*) single tailed).

(*iii*) $H_1: \mu < \mu_0$ (left tailed alternative hypothesis (*or*) single tailed).

Hence alternative hypothesis helps to know whether the test is two tailed test or one tailed test.

6.6. CRITICAL REGION

A region corresponding to a statistic t, in the sample space S which amounts to rejection of the null hypothesis H_0 is called as **critical region** or **region of rejection**. The region of the sample space S which amounts to the acceptance of H_0 is called *acceptance region*.

6.7. LEVEL OF SIGNIFICANCE

The probability of the value of the variate falling in the critical region is known as level of significance.

The probability α that a random value of the statistic t belongs to the critical region is known as the level of significance.

$$P(t \in \omega \mid H_0) = \alpha$$

i.e., the level of significance is the size of the type I error or the maximum producer's risk.

6.8. ERRORS IN SAMPLING

The main aim of the sampling theory is to draw a valid conclusion about the population parameters. On the basis of the sample results. In doing this we may commit the following two types of errors:

Type I. Error. When H_0 is true, we may reject it.

P(Reject H_0 when it is true) = P(Reject H_0/H_0) = α

α is called the size of the type I error also referred to as **producer's risk.**

Type II. Error. When H_0 is wrong we may accept it P(Accept H_0 when it is wrong) = P(Accet H_0/H_1) = β. β is called the size of the type II error, also referred to as **consumer's risk.**

Critical values or significant values

The values of the test statistic which separates the critical region and acceptance region is called the **critical values** or **significant value.**

This value is dependent on (*i*) the level of significance used and (*ii*) the alternative hypothesis, whether it is one tailed or two tailed.

For larger samples corresponding to the statistic t, the variable $z = \dfrac{t - E(t)}{S.E.(t)}$ is normally distributed with mean 0 and variance 1. The value of z given above under the null hypothesis is known as **test statistic.**

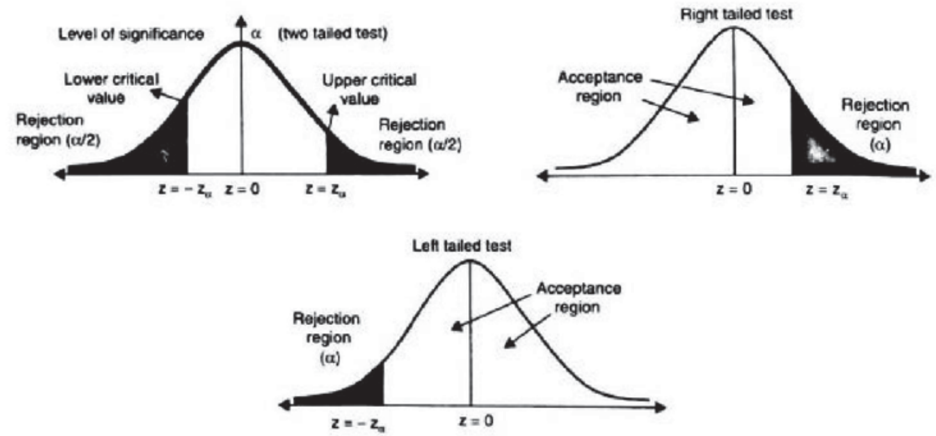
The critical value of z_α of the test statistic at level of significance α for a two tailed test is given by

$$p(|z| > z_\alpha) = \alpha \qquad ...(1)$$

i.e., z_α is the value of z so that the total area of the critical region on both tails is α. Since the normal curve is a symmetrical, from equation (1), we get

$$p(z > z_\alpha) + p(z < -z_\alpha) = \alpha;\ i.e.,\ 2p(z > z_\alpha) = \alpha;\ p(z > z_\alpha) = \alpha/2$$

i.e., the area of each tail is $\alpha/2$.



The critical value z_α is that value such that the area to the right of z_α is $\alpha/2$ and the area to the left of $-z_\alpha$ is $\alpha/2$.

In the case of one tailed test

$$p(z > z_\alpha) = \alpha \text{ if it is right tailed; } p(z < -z_\alpha) = \alpha \text{ if it is left tailed.}$$

The critical value of z for a single—tailed test (right or left) at level of significance α is same as the critical value of z for two-tailed test at level of significance 2α.

Using the equation, also using the normal tables, the critical value of z at different level of significance (α) for both single tailed and two tailed test are calculated and listed below. The equations are

$$p(|z| > z_\alpha) = \alpha;\ p(z > z_\alpha) = \alpha;\ p(z < -z_\alpha) = \alpha$$

	Level of significance		
	1% (0.01)	*5% (0.05)*	*10% (0.1)*
Two tailed test	$\lvert z_\alpha \rvert = 2.58$	$\lvert z_\alpha \rvert = 1.966$	$\lvert z_\alpha \rvert = 0.645$
Right tailed	$z_\alpha = 2.33$	$z_\alpha = 1.645$	$z_\alpha = 1.28$
Left tailed	$z_\alpha = -2.33$	$z_\alpha = -1.645$	$z_\alpha = -1.28$

Note. The following steps may be adopted in testing of statistical hypothesis:

Step 1: Null hypothesis. Set up H_0 is clear terms.

Step 2: Alternative hypothesis. Set up H_1, so that we could decide whether we should used one-tailed test or two tailed test.

Step 3: Level of significance. Select the appropriate level of significance in advance depending on the reliability of the estimates.

Step 4: Test statistic. Compute the test statistic $z = \dfrac{t - E(t)}{S.E.(t)}$ under the null hypothesis.

Step 5: Conclusion. Compare the computed value of z with the critical value z_α at level of significance (α).

If $|z| > z_\alpha$, we reject H_0 and conclude that there is significant difference. If $|z| < z_\alpha$, we accept H_0 and conclude that there is no significant difference.

TEST OF SIGNIFICANCE FOR LARGE SAMPLES

If the sample size $n > 30$, the sample is taken as large sample. For such sample we apply normal test, as Binomial, Poisson, chisquare etc. are closely approximated by normal distributions assuming the population as normal.

Under large sample test, the following are the important tests to test the significance.

1. *Testing of significance for single proportion.*
2. *Testing of significance for difference of proportions.*
3. *Testing of significance for single mean.*
4. *Testing of significance for difference of means.*

6.9. TESTING OF SIGNIFICANCE FOR SINGLE PROPORTION

This test is used to find the significant difference between proportion of the sample and the population.

Let X be the number of successes in n independent trials with constant probability P of success for each trial.

$$E(X) = nP;\ V(X) = nPQ;\ Q = 1 - P = \text{Probability of failure.}$$

Let $p = X/n$ called the observed proportion of success.

$$E(p) = E(X/n) = \frac{1}{n} E(X) = \frac{nP}{n} = P;\ E(p) = P$$

$$V(p) = V(X/n) = \frac{1}{n^2} v(X) = \frac{n(PQ)}{n^2} = PQ/n$$

$$\text{S.E.}(p) = \sqrt{\frac{PQ}{n}};\ z = \frac{p - E(p)}{\text{S.E.}(p)} = \frac{p - P}{\sqrt{PQ/n}} \sim N(0, 1)$$

This z is called test statistic which is used to test the significant difference of sample and population proportion.

Note 1. The probable limit for the observed proportion of successes are $P \pm 3\sqrt{PQ/n}$

2. If P is not known, the limits for the proportion in the population are $P \pm z_\alpha \sqrt{PQ/n}$, $Q = 1 - p$ where z_α is the significant value of z at level of significance α.

ILLUSTRATIVE EXAMPLES

Example 1. *A coin was tossed 400 times and the head turned up 216 times. Test the hypothesis that the coin is unbiased.* (V.T.U. 2000)

Sol. H_0: The coin is unbiased *i.e.*, $P = 0.5$.

H_1: The coin is not unbiased. (biased); $P \neq 0.5$

Here $n = 400$; X = No. of success = 216

$$p = \underline{\text{proportion of success in the sample}} = \frac{X}{n} = \frac{216}{400} = 0.54$$

population proportion $= 0.5 = P$; $Q = 1 - P = 1 - 0.5 = 0.5$

under H_0, test statistic $z = \dfrac{p - P}{\sqrt{PQ/n}}$

$$|z| = \left| \frac{0.54 - 0.5}{\sqrt{\frac{0.5 \times 0.5}{400}}} \right| = 1.6$$

we use two tailed test.

Conclusion. Since $|z| = 1.6 < 1.96$

i.e., $|z| < z_\alpha$, z_α is the significant value of z at 5% level of significance.

i.e., the coin is unbiased is $P = 0.5$.

Example 2. *A cubical die is thrown 9000 times and a throw of 3 or 4 is observed 3240 times. Show that the die cannot be regarded as an unbiased one and find the extreme limits between which the probability of a throw of 3 or 4 lies.* (M.D.U. May 2009)

Sol. Here $n = 9000$

P = probability of success (*i.e.*, getting 3 or 4 on die)

$P = 2/6 = 1/3$, $Q = 1 - 1/3 = 2/3$

$$p = \frac{X}{m} = \frac{3240}{9000} = 0.36$$

H_0: is unbiased, *i.e.*, $P = 1/3$

H_1: $P \neq 1/3$ (two tailed test)

The test statistic $$z = \frac{p - P}{\sqrt{\frac{PQ}{n}}} = \frac{0.36 - 0.33}{\sqrt{\frac{1}{3} \times \frac{2}{3} \times \frac{1}{9000}}} = 0.03496$$

$$|z| = 0.03496 < 1.96$$

Conclusion. Accept the hypothesis

As $|z| < z_\alpha$, z_α is the tabulated value of z at 5% level of significance.

$\because$ H_0 is accepted, we conclude that the die is unbiased.

To find 95% confidence limits of the proprtion.

It is given by $P \pm z_\alpha \sqrt{\frac{PQ}{n}}$

i.e., $0.33 \pm 1.96 \sqrt{\frac{0.33 \times 0.67}{9000}} = 0.33 \pm 0.0097 = 0.3203$ and 0.3397.

Example 3. *A manufacturer claims that only 4% of his products supplied by him are defective. A random sample of 600 products contained 36 defectives. Test the claim of the manufacturer.*

Sol. (*i*) P = observed proportion of success.

i.e., P = proportion of defective in the sample $= \frac{36}{600} = 0.06$

p = proportion of defectives in the population = 0.04

H_0: $p = 0.04$ is true.

i.e., the claim of the manufacturer is accepted.

H_1: (*i*) $P \neq 0.04$ (two teiled test)

(*ii*) If we want to reject, only if $p > 0.04$ then (right tailed).

Under H_0, $$z = \frac{p - P}{\sqrt{PQ/n}} = \frac{0.06 - 0.04}{\sqrt{\frac{0.04 \times 0.96}{600}}} = 2.5.$$

Conclusion. Since $|z| = 2.5 > 1.96$, we reject the hypothesis H_0 at 5% level of significance two tailed.

If H_1 is taken as $p > 0.04$ we apply right tailed test.

$|z| = 2.5 > 1.645$ (z_α) we reject the null hypothesis here also.

In both cases, manufacturer's claim is not acceptable.

Example 4. *A machine is producing bolts of which a certain fraction is* ***defective.*** *A random sample of 400 is taken from a large batch and is found to contain 30* ***defective bolt.*** *Does this indicate that the proportion of defectives is larger than that claimed by the* ***manufacturer*** *where the manufacturer claims that only 5% of his product are defective. Find* ***95% confidence*** *limits of the proportion of defective bolts in batch.*

Sol. Null hypothesis. H_0: The manufacturer claim is accepted *i.e.*, $P = \frac{5}{100} = 0.05$

$$Q = 1 - P = 1 - 0.05 = 0.95$$

Alternative hypothesis. $p > 0.05$ (Right tailed test).

p = observed proportion of sample $= \dfrac{30}{400} = 0.075$

Under H_0, the test statistic $z = \dfrac{p - P}{\sqrt{PQ/n}}$ $\therefore$ $z = \dfrac{0.075 - 0.05}{\sqrt{\dfrac{0.05 \times 0.95}{400}}} = 2.2941.$

Conclusion. The tabulated value of z at 5% level of significance for right tailed test,

$z_\alpha = 1.645$. Since $|z| = 2.2941 > 1.645$,

H_0 is rejected at 5% level of significance. *i.e.*, the proportion of defective is larger than the manufacturer claim.

To find 95% confidence limits of the proportion.

It is given by $P \pm z_\alpha \sqrt{PQ/n}$

$$0.05 \pm 1.96 \sqrt{\frac{0.05 \times 0.95}{400}} = 0.05 \pm 0.02135 = 0.02865 \text{ and } 0.07136.$$

Hence 95% confidence limits for the proportion of defective bolt are (0.07136, 0.02865).

Example 5. *A bag contains defective article, the exact number of which is not known. A sample of 100 from the bag gives 10 defective articles. Find the limits for the proportion of defective articles in the bag.*

Sol. Here p = proportion of defective articles $= \dfrac{10}{100} = 0.1;\ q = 1 - p = 1 - 0.1 = 0.9$

Since the confidence limits is not given, we assume it is 95%.

$\therefore$ level of significance is 5% $z_\alpha = 1.96$.

Also the proportion of population P is not given. To get the confidence limit, we use p and it is given by $P \pm z_\alpha \sqrt{pq/n} = 0.1 \pm 1.96 \sqrt{\dfrac{0.1 \times 0.9}{100}} = 0.1 \pm 0.0588 = 0.0412$ and 0.1588

Hence 95% confidence limits for defective article in the bag are (0.0412, 0.1588).

EXERCISE 6.1

1. A sample of size of 600 persons selected at random from a large city shows that the percentage of males in the sample is 53. It is believed that the ratio of males to the total population in the city is 0.5. Test whether the belief is confirmed by the observation.
2. In a city a sample of 1000 people were taken and out of them 540 are vegetarian and the rest are non-vegetarian. Can we say that the both habits of eating (Vegetation or non vegetation) are equally popular in the city at (*i*) 1% level of significance (*ii*) 5% level of significance?
3. 325 men out of 600 men chosen from a big city were found to be smokers. Does this information support the conclusion that the majority of men in the city are smokers.
4. A random sample of 500 bolts was taken from a large consignment and 65 were found to be defective. Find the percentage of defective bolts in the consignment.
5. In a hospital 475 female and 525 male babies were burn in a week. Do these figure confirm the hypothesis that males and females are born in equal number?
6. 400 apples are taken at random from a large basket and 40 are found to be bad. Estimate the proportion of bad apples in the basket and assign limits within which the percentage most probably lies.

7. A manufacturer claims that only 10% of the articles produced are below the standard quality. Out of a random inspection of 300 articles, 37 are found to be of poor quality. Test the manufacturer's claim at 5% level of significance.

Answers

1. H_0 accepted at 5% level	2. H_0 rejected at 5% level, accepted at 1% level	
3. H_0 rejected at 5% level	4. Between 17.51 and 8.49	5. H_0 accepted at 5% level
6. 8.5 : 11.5	7. H_0 accepted at 5% level.	

6.10. TEST OF DIFFERENCE BETWEEN PROPORTIONS

Consider two samples X_1 and X_2 of sizes n_1 and n_2 respectively taken from two different populations. To test the significance of the difference between the sample proportion p_1 and p_2. The test statistic under the null hypothesis H_0, that there is no significant difference between the two sample proportion, we have

$$z = \frac{p_1 - p_2}{\sqrt{PQ\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}, \qquad \text{where } P = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} \text{ and } Q = 1 - P.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Before an increase in excise duty on tea, 800 people out of a sample of 1000 persons were found to be tea drinkers. After an increase in the duty, 800 persons were known to be tea drinkers in a sample of 1200 people. Do you think that there has been a significant decrease in the consumption of tea after the increase in the excise duty?*

Sol. Here $n_1 = 800,\ n_2 = 1200$

$$p_1 = \frac{X_1}{n_1} = \frac{800}{1000} = \frac{4}{5};\ p_2 = \frac{X_2}{n_2} = \frac{800}{1200} = \frac{2}{3}$$

$$P = \frac{p_1 n_1 + p_2 n_2}{n_1 + n_2} = \frac{X_1 + X_2}{n_2 + n_2} = \frac{800 + 800}{1000 + 1200} = \frac{8}{11};\ Q = \frac{3}{11}$$

Null hypothesis $\mathbf{H_0}$: $p_1 = p_2$ *i.e.*, there is no significant difference in the consumption of tea before and after increase of excise duty.

H_1: $p_1 > p_2$ (right tailed test)

The test statistic $z = \dfrac{p_1 - p_2}{\sqrt{PQ\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}} = \dfrac{0.8 - 0.6666}{\sqrt{\dfrac{8}{11} \times \dfrac{3}{11}\left(\dfrac{1}{1000} + \dfrac{1}{1200}\right)}} = 6.842.$

Conclusion. Since the calculated value of $|z| > 1.645$ also $|z| > 2.33$, both the significant value of z at 5% and 1% level of significance. Hence H_0 is rejected *i.e.*, there is a significant decrease in the consumption of tea due to increase in excise duty.

Example 2. *A machine produced 16 defective articles in a batch of 500. After overhauling it produced 3 defectives in a batch of 100. Has the machine improved?*

(M.D.U. May 2007)

Sol. $p_1 = \dfrac{16}{500} = 0.032;\ n_1 = 500$ | $p_2 = \dfrac{3}{100} = 0.03;\ n_2 = 100$

Null hypothesis H_0: The machine has not improved due to overhauling. $p_1 = p_2$.

$$H_1: p_1 > p_2 \text{ (right tailed)} \quad \therefore \quad P = \frac{p_1n_1 + p_2n_2}{n_1 + n_2} = \frac{19}{600} \equiv 0.032$$

Under H_0, the test statistic

$$z = \frac{p_1 - p_2}{\sqrt{PQ\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}} = \frac{0.032 - 0.03}{\sqrt{(0.032)(0.968)\left(\dfrac{1}{500} + \dfrac{1}{100}\right)}} = 0.104.$$

Conclusion. The calculated value of $|z| < 1.645$, the significant value of z at 5% level of significance. H_0 is accepted. *i.e.*, the machine has not improved due to overhauling.

Example 3. *In two large populations there are 30% and 25% respectively of fair haired people. Is this difference likely to be hidden in samples of 1200 and 900 respectively from the two populations.* (M.D.U. Dec. 2010)

Sol. p_1 = proportion of fair haired people in the first population = 30% = 0.3;
$p_2 = 25\% = 0.25;\ Q_1 = 0.7,\ Q_2 = 0.75$

H_0: Sample proportions are equal *i.e.*, the difference in population proportions is likely to be hidden in sampling.

$H_1: p_1 \neq p_2$

$$z = \frac{P_1 - P_2}{\sqrt{\left(\dfrac{P_1Q_1}{n_1} + \dfrac{P_2Q_2}{n_2}\right)}} = \frac{0.3 - 0.25}{\sqrt{\dfrac{0.3 \times 0.7}{1200} + \dfrac{0.25 \times 0.75}{900}}} = 2.5376.$$

Conclusion. Since $|z| > 1.96$, the significant value of z at 5% level of significance, H_0 is rejected. However $|z| < 2.58$, the significant value of z at 1% level of significance. H_0 is accepted. At 5% level these samples will reveal the difference in the population proportions.

Example 4. *500 articles from a factory are examined and found to be 2% defective. 800 similar articles from a second factory are found to have only 1.5% defective. Can it reasonably be concluded that the product of the first factory are inferior to those of second?*

Sol. $n_1 = 500,\ n_2 = 800$

p_1 = proportion of defective from first factory = 2% = 0.02

p_2 = proportion of defective from second factory = 1.5% = 0.015

H_0: There is no significant difference between the two products *i.e.*, the products do not differ in quality.

$H_1: p_1 < p_2$ (one tailed test)

Under H_0, $$z = \frac{p_1 - p_2}{\sqrt{PQ\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}$$

$$P = \frac{n_1p_1 + n_2p_2}{n_1 + n_2} = \frac{0.02(500) + (0.015)(800)}{500 + 800} = 0.01692;\ Q = 1 - P = 0.9830$$

$$z = \frac{0.02 - 0.015}{\sqrt{0.01692 \times 0.983\left(\dfrac{1}{500} + \dfrac{1}{800}\right)}} = 0.68$$

Conclusion. As $|z| < 1.645$, the significant value of z at 5% level of significance. H_0 is accepted. *i.e.*, the products do not differ in quality.

EXERCISE 6.2

1. Random sample of 400 men and 600 women were asked whether they would to have a school near their residence. 200 men and 325 women were in favour of proposal. Test the hypothesis that the proportion of men and women in favour of the proposal are same at 5% level of significance.
2. In a town A, there were 956 births of which 52.5% was males while in towns A and B combined, this proportion in total of 1406 births was 0.496. Is there any significant difference in the proportion of male births in the two towns?
3. In a referendum submitted to the students body at a university, 850 men and 560 women voted. 500 men and 320 women voted yes. Does this indicate a significant difference of opinion between men and women on this matter at 1% level?
4. A manufacturing firm claims that its brand A product outsells its brand B product by 8%. If it is found that 42 out of a sample of 200 person prefer brand A and 18 out of another sample of 100 person prefer brand B. Test whether the 8% difference is a valid claim.

Answers

1. H_0: accepted **2.** H_0: Rejected **3.** H_0: accepted **4.** H_0: accepted.

6.11. TEST OF SIGNIFICANCE FOR SINGLE MEAN

To test whether the difference between sample mean and population mean is significant or not.

Let $X_1, X_2, \ldots\ldots, X_n$ be a random sample of size n from a large population $X_1, X_2, \ldots\ldots, X_N$ of size N with mean μ and variance σ^2. $\therefore$ the standard error of mean of a random sample of size n from a population with variance σ^2 is $\sigma/\sqrt{n}$.

To test whether the given sample of size n has been drawn from a population with mean μ *i.e.*, to test whether the difference between the sample mean and population mean is significant or not. Under the null hypothesis that there is no difference between the sample mean and population mean.

The test statistic is $z = \dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}}$, where σ is the standard deviation of the population.

If σ is not known, we use the test statistic $z = \dfrac{\bar{X} - \mu}{s/\sqrt{n}}$, where s is the standard deviation of the sample.

Note. If the level of significance is α and z_α is the critical value $-z_\alpha < |z| = \left|\dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}}\right| < z_\alpha$

The limits of the population mean μ are given by $\bar{x} - z_\alpha \dfrac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_\alpha\, \sigma/\sqrt{n}$.

At 5% of level of significance, 95% confidence limits are $\bar{x} - 1.96 \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + 1.96 \frac{\sigma}{\sqrt{n}}$

At 1% level of significance, 99% confidence limits are $\bar{x} - 2.58 \frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + 2.58 \frac{\sigma}{\sqrt{n}}$.

These limits are called confidence limits or fiducial limits.

ILLUSTRATIVE EXAMPLES

Example 1. *A normal population has a mean of 6.8 and standard deviation of 1.5. A sample of 400 members gave a mean of 6.75. Is the difference significant?*

Sol. H_0: There is no significant difference between $\bar{x}$ and μ.

H_1: There is significant difference between $\bar{x}$ and μ.

Given $\mu = 6.8$, $\sigma = 1.5$, $\bar{x} = 6.75$ and $n = 400$

$$|z| = \left|\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right| = \left|\frac{6.75-6.8}{1.5/\sqrt{900}}\right| = |-0.67| = 0.67$$

Conclusion. As the calculated value of $|z| < z_\alpha = 1.96$ at 5% level of significance, H_0 is accepted *i.e.*, there is no significant difference between $\bar{x}$ and μ.

Example 2. *A random sample of 900 members has a mean 3.4 cms. Can it be reasonably regarded as a sample from a large population of mean 3.2 cms and S.D. 2.3 cms?*

Sol. Here $n = 900$, $\bar{x} = 3.4$, $\mu = 3.2$, $\sigma = 2.3$

H_0: Assume that the sample is drawn from a large population with mean 3.2 and S.D. = 2.3

H_1: $\mu \neq 3.25$ (Apply two tailed test)

Under H_0; $z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} = \frac{3.4-3.2}{2.3/\sqrt{900}} = 0.261.$

Conclusion. As the calculated value of $|z| = 0.261 < 1.96$ the significant value of z at 5% level of significance. H_0 is accepted *i.e.*, the sample is drawn from the population with mean 3.2 and S.D. = 2.3.

Example 3. *The mean weight obtained from a random sample of size 100 is 64 gms. The S.D. of the weight distribution of the population is 3 gms. Test the statement that the mean weight of the population is 67 gms at 5% level of significance. Also set up 99% confidence limits of the mean weight of the population.*

Sol. Here $n = 100$, $\mu = 67$, $\bar{x} = 64$, $\sigma = 3$

H_0: There is no significant difference between sample and population mean.

i.e., $\mu = 67$, the sample is drawn from the population with $\mu = 67$

H_1: $\mu \neq 67$ (Two tailed test)

Under H_0, $z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} = \frac{64-67}{3/\sqrt{100}} = -10 \quad \therefore \quad |z| = 10.$

Conclusion. Since the calculated value of $|z| > 1.96$, the significant value of z at 5% level of significance. H_0 is rejected *i.e.*, the sample is not drawn from the population with mean 67.

To find 99% confidence limits. It is given by $\bar{x} \pm 2.58\, \sigma/\sqrt{n} = 64 \pm 2.58\; 3/\sqrt{100} = 64.774$, 63.226.

Example 4. *The average marks in Mathematics of a sample of 100 students was 51 with a S.D. of 6 marks. Could this have been a random sample from a population with average marks 50?* (M.D.U. Dec. 2011)

Sol. Here $n = 100$, $\bar{x} = 51$, $s = 6$, $\mu = 50$; σ is unknown.

H_0: The sample is drawn from a population with mean 50, $\mu = 50$

H_1: $\mu \neq 50$

Under H_0, $z = \frac{\bar{x}-\mu}{s/\sqrt{n}} = \frac{51-50}{6/\sqrt{100}} = \frac{10}{6} = 1.6666$

Conclusion. Since $|z| = 1.666 < 1.96$, z_α the significant value of z at 5% level of significance, H_0, is accepted *i.e.*, the sample is drawn from the population with mean 50.

EXERCISE 6.3

1. A sample of 1000 students from a university was taken and their average weight was found to be 112 pounds with a S.D. of 20 pounds. Could the mean weight of students in the population be 120 pounds?
2. A sample of 400 male students is found to have a mean height of 160 cms. Can it be reasonably regarded as a sample from a large population with mean height 162.5 cms and standard deviation 4.5 cms.
3. A random sample of 200 measurements from a large population gave a mean value of 50 and a S.D. of 9. Determine 95% confidence interval for the mean of population.
4. The guaranteed average life of certain type of bulbs is 1000 hours with a S.D. of 125 hours. It is decided to sample the output so as to ensure that 90% of the bulbs do not fall short of the guaranteed average by more than 2.5%. What must be the minimum size of the sample.
5. The heights of college students in a city are normally distributed with S.D. 6 cms. A sample of 1000 students has mean height 158 cms. Test the hypothesis that the mean height of college students in the city is 160 cms.

Answers

1. H_0 is rejected **2.** H_0 accepted **3.** 48.8 and 51.2 **4.** $n = 4$

5. H_0 rejected both at 1% to 5% level of significance.

6.12. TEST OF SIGNIFICANCE FOR DIFFERENCE OF MEANS OF TWO LARGE SAMPLES

Let $\bar{x}_1$ be the mean of a sample of size n_1 from a population with mean μ_1, and variance σ_1^2. Let $\bar{x}_2$ be the mean of an independent sample of size n_2 from another population with mean μ_2 and variance σ_2^2. The test statistic is given by $z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$.

Under the null hypothesis that the samples are drawn from the same population where $\sigma_1 = \sigma_2 = \sigma$ *i.e.*, $\mu_1 = \mu_2$ the test statistic is given by $z = \frac{\bar{x}_1 - \bar{x}_2}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$.

Note 1. If σ_1, σ_2 are not known and $\sigma_1 \neq \sigma_2$ the test statistic in this case is $z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}}$.

Note 2. If σ is not known and $\sigma_1 = \sigma_2$

We use $\sigma^2 = \dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2}$ to calculate σ; $z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2}\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}$.

ILLUSTRATIVE EXAMPLES

Example 1. *The average income of persons was ₹ 210 with a S.D. of ₹ 10 in sample of 100 people of a city. For another sample of 150 persons, the average income was ₹ 220 with S.D. of ₹ 12. The S.D. of incomes of the people of the city was ₹ 11. Test whether there is any significant difference between the average incomes of the localities.*

Sol. Here $n_1 = 100, n_2 = 150, \bar{x}_1 = 210, \bar{x}_2 = 220, s_1 = 10, s_2 = 12.$

Null hypothesis. The difference is not significant. *i.e.* there is no difference between the incomes of the localities.

$$H_0: \bar{x}_1 = \bar{x}_2, \ H_1: \bar{x}_1 \neq \bar{x}_2$$

Under H_0, $z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}} = \dfrac{210 - 220}{\sqrt{\dfrac{10^2}{100} + \dfrac{12^2}{150}}} = -7.1428 \quad \therefore \quad |z| = 7.1428.$

Conclusion. As the calculated value of $|z| > 1.96$ the significant value of z at 5% level of significance, H_0 is rejected *i.e.*, there is significant difference between the average incomes of the localities.

Example 2. *Intelligent tests were given to two groups of boys and girls.*

	Mean	*S.D.*	*size*
Girls	*75*	*8*	*60*
Boys	*73*	*10*	*100*

Examine if the difference between mean scores is significant. (M.D.U. Dec. 2011)

Sol. Null hypothesis H_0: There is no significant difference between mean scores *i.e.*, $\bar{x}_1 = \bar{x}_2$

$$H_1: \bar{x}_1 \neq \bar{x}_2$$

Under the null hypothesis $z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}} = \dfrac{75 - 73}{\sqrt{\dfrac{8^2}{60} + \dfrac{10^2}{100}}} = 1.3912.$

Conclusion. As the calculated value of $|z| < 1.96$, the significant value of z at 5% level of significance, H_0 is accepted. *i.e.*, there is no significant difference between mean scores.

Example 3. *For sample I, $n_1 = 1000$, $\Sigma x = 49{,}000$, $\Sigma(x - \bar{x})^2 = 7{,}84{,}000$.*

For sample II, $n_2 = 1{,}500$, $\Sigma x = 70{,}500$, $\Sigma(x - \bar{x})^2 = 24{,}00{,}000$. Discuss the significance of the difference of the sample means.

Sol. Null hypothesis H_0: There is no significant difference between the sample means.

$$H_0: \bar{x}_1 = \bar{x}_2; \ H_1: \bar{x}_1 \neq \bar{x}_2$$

To calculate sample variance

$$s_1^2 = \frac{1}{n_1}\Sigma(X_1 - \bar{X}_1)^2 = \frac{784000}{1000} = 784$$

$$s_2^2 = \frac{1}{n_2}\Sigma(X_2 - \bar{X}_2)^2 = \frac{1}{1500}(2400000) = 1600$$

$$\bar{x}_1 = \frac{\Sigma x_1}{n_1} = \frac{49000}{1000} = 49; \ \bar{x}_2 = \frac{\Sigma x_2}{n_2} = \frac{70500}{1500} = 47$$

Under the null hypothesis, the test statistic

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}} = \frac{49 - 47}{\sqrt{\dfrac{784}{1000} + \dfrac{1600}{1500}}} = 1.470.$$

Conclusion. As the calculated value of $|z| = 1.47 < 1.96$, the significant value of z at 5% level of significance, H_0 is accepted. *i.e.*, there is no significant difference between the sample means.

Example 4. *From the data given below, compute the standard error of the difference of the two sample means and find out if the two means significantly differ at 5% level of significance.*

	No. of items	*Mean*	*S.D.*
Group I	*50*	*181.5*	*3.0*
Group II	*75*	*179*	*3.6*

Sol. Null hypothesis H_0: There is no significant difference between the samples.

$$\bar{x}_1 = \bar{x}_2; \ H_1: \bar{x}_1 \neq \bar{x}_2$$

Under H_0, $z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}} = \dfrac{181.5 - 179.0}{\sqrt{\dfrac{9}{50} + \dfrac{(3.6)^2}{75}}} = 4.2089.$

Conclusion. As $|z| >$ the tabulated value of z at 5% level of significance H_0 is rejected *i.e.*, there is significant difference between the samples.

Example 5. *A random sample of 200 villages from Coimbatore district gives the mean population per village at 485 with a S.D. of 50. Another random sample of the same size from the same district gives the mean population per village at 510 with a S.D. of 40. Is the difference between the mean values given by the two samples statistically significant? Justify your answer.*

Sol. Here $n_1 = 200, n_2 = 250, \bar{x}_1 = 485, \bar{x}_2 = 510, s_1 = 50, s_2 = 40$

Null hypothesis H_0: There is no significant difference between the mean values (*i.e.*,) $\bar{x}_1 = \bar{x}_2$; H: $\bar{x}_1 \neq \bar{x}_2$ (Two tailed test)

Under H_0, the test statistic is given by $z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}} = \dfrac{485 - 510}{\sqrt{\dfrac{50^2}{200} + \dfrac{40^2}{200}}} = -5.52$

$\therefore \quad |z| = 5.52$

Conclusion. As the calculated value of $|z| > 1.96$, the significant value of z at 5% level of significance, H_0 is rejected *i.e.*, there is significant difference between the mean values of two sample.

EXERCISE 6.4

1. Intelligence tests on two groups of boys and girls gave the following results. Examine if the difference is significant.

	Mean	*S.D.*	*Size*
Girls	*70*	*10*	*70*
Boys	*75*	*11*	*100*

2. (*a*) Two random samples of sizes 1000 and 2000 farms gave an average yield of 2000 kg and 2050 kg respectively. The variance of wheat farms in the country may be taken as 100 kg. Examine whether the two samples differ significantly in yield?

   (*b*) The means of simple samples of sizes 1000 and 2000 are 67.5 and 68.0 cm respectively. Can the samples be regarded as drawn from the same population of S.D. 2.5 cm? *(M.D.U. May 2011)*

3. A simple sample of heights of 6400 Englishmen has a mean of 170 cm and a S.D. of 6.4 cm, while a simple sample of heights of 1600 Americans has a mean of 172 cm and a S.D. of 6.3 cm. Does the data indicate that Americans are, on the average, taller than Englishmen? *(M.D.U. May 2009)*

4. In a survey of buying habits, 400 women shoppers are chosen at random in supermarket A. Their average weekly food expenditure is ₹ 250 with a S.D. of ₹ 40. For 500 women shoppers chosen at supermarket B, the average weekly food expenditure is ₹ 220 with a S.D. of ₹ 45 test at 1% level of significance whether the average food expenditures of the two groups are equal.

5. The number of accident per day were studied for 144 days in town A and for 100 days in town B and the following information was obtained.

	Mean number of accident	*S.D.*
Town A	4.5	1.2
Town B	5.4	1.5

   Is the difference between the mean accidents of the two towns statistically significant?

6. An examination was given to 50 students of a college A and to 60 students of college B. For A, the mean grade was 75 with S.D. of 9 and for B, the mean grade was 79 with S.D. of 7. Is there any significant difference between the performance of the students of college A and those of college B?

7. A random sample of 200 measurements from a large population gave a mean value of 50 and S.D. of 9. Determine the 95% confidence interval for the mean of the population?

8. The means of two large samples of 1000 and 2000 members are 168.75 cms and 170 cms respectively. Can the samples be regarded as drawn from the same population of standard deviation 6.25 cms?

Answers

1. No significant difference
2. (*a*) Highly significant (*b*) No
3. Highly significant
4. Highly significant
5. Highly significant
6. Not significant
7. 49.584, 50.416
8. Not significant

6.13. TEST OF SIGNIFICANCE FOR THE DIFFERENCE OF STANDARD DEVIATIONS

If s_1 and s_2 are the standard deviations of two independent samples then under the null hypothesis $H_0: \sigma_1 = \sigma_2$, *i.e.*, the sample standard deviations don't differ significantly, the statistic

$$z = \frac{s_1 - s_2}{\sqrt{\dfrac{\sigma_1^2}{2n_1} + \dfrac{\sigma_2^2}{2n_2}}}, \text{ where } \sigma_1 \text{ and } \sigma_2 \text{ are population standard deviations.}$$

when population standard deviations are not known then $z = \dfrac{s_1 - s_2}{\sqrt{\dfrac{s_1^2}{2n_1} + \dfrac{s_2^2}{2n_2}}}$.

ILLUSTRATIVE EXAMPLES

Example 1. *Random samples drawn from two countries gave the following data relating to the heights of adult males.*

	Country A	*Country B*
Mean height (in inches)	*67.42*	*67.25*
Standard deviation	*2.58*	*2.50*
Number in samples	*1000*	*1200*

(*i*) *Is the difference between the means significant?*

(*ii*) *Is the difference between the standard deviations significant?*

Sol. Given: $n_1 = 1000$, $n_2 = 1200$, $\bar{x}_1 = 67.42$, $\bar{x}_2 = 67.25$, $s_1 = 2.58$, $s_2 = 2.50$

Since the samples size are large we can take $\sigma_1 = s_1 = 2.58$; $\sigma_2 = s_2 = 2.50$

(*i*) **Null Hypothesis:** $H_0 = \mu_1 = \mu_2$ *i.e.*, sample means do not differ significantly.

Alternative hypothesis: $H_1: \mu_1 \neq \sigma_2$ (two tailed test)

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}} = \frac{67.42 - 67.25}{\sqrt{\dfrac{(2.58)^2}{1000} + \dfrac{(2.50)^2}{1200}}} = 1.56$$

since $|z| < 1.96$ we accept the null hypothesis at 5% level of significance.

(*ii*) We set up the null hypothesis.

$H_0: \sigma_1 = \sigma_2$ *i.e.*, the sample S.D.'s do not differ significantly.

Alternative Hypothesis: $H_1 = \sigma_1 \neq \sigma_2$ (two tailed)

∴ The test statistic is given by

$$z = \frac{s_1 - s_2}{\sqrt{\frac{\sigma_1^2}{2n_1} + \frac{\sigma_2^2}{2n_2}}} = \frac{s_1 - s_2}{\sqrt{\frac{s_1^2}{2n_1} + \frac{s_2^2}{2n_2}}} \quad (\because \ \sigma_1 = s_1, \sigma_2 = s_2 \text{ for large samples})$$

$$= \frac{2.58 - 2.50}{\sqrt{\frac{(2.58)^2}{2 \times 1000} \times \frac{(2.50)^2}{2 \times 1200}}} = \frac{0.08}{\sqrt{\frac{6.6564}{2000} + \frac{6.25}{2400}}} = 1.0387$$

Since $|z| < 1.96$ we accept the null hypothesis at 5% level of significance.

Example 2. *Intelligence test of two groups of boys and girls gives the following results:*

Girls	*mean = 84*	*S.D. = 10*	*N = 121*
Boys	*mean = 81*	*S.D. = 12*	*N = 81*

(*a*) *Is the difference in mean scores significant?*

(*b*) *Is the difference between the standard deviations significant?*

Sol. Given: $n_1 = 121$, $n_2 = 81$, $\bar{x}_1 = 84$, $\bar{x}_2 = 81$, $s_1 = 10$, $s_2 = 12$

(*a*) **Null hypothesis:** $H_0 = \mu_1 = \mu_2$ *i.e.*, sample means do not differ significantly.

Alternative hypothesis: $H_1 = \mu_1 \neq \mu_2$ (two tailed)

The test statistic is $$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{84 - 81}{\sqrt{\frac{(10)^2}{121} + \frac{(12)^2}{81}}} = 0.1859$$

Since $|z| < 1.96$ we accept the null hypothesis at 5% level of significance.

(*b*) We set up the null hypothesis $H_0 = \sigma_1 = \sigma_2$ *i.e.*, the sample S.D.'s do not differ significantly. Alternative hypothesis: $H_1 = \sigma_1 \neq \sigma_2$ (two tailed)

The test statistic is $$z = \frac{s_1 - s_2}{\sqrt{\frac{\sigma_1^2}{2n_1} + \frac{\sigma_2^2}{2n_2}}} = \frac{s_1 - s_2}{\sqrt{\frac{s_1^2}{2n_1} + \frac{s_2^2}{2n_2}}}$$

$$(\because \ \sigma_1 = s_1, \sigma_2 = \sigma_2 \text{ for large samples})$$

$$= \frac{10 - 12}{\sqrt{\frac{100}{2 \times 121} + \frac{144}{2 \times 81}}} = -1.7526 \quad \therefore \quad |z| = 1.7526$$

since $|z| = 1.75 < 1.96$ we accept the null hypothesis at 5% level of significance.

EXERCISE 6.5

1. The mean yield of two sets of plots and their variability are as given examine.
   (*i*) Whether the difference in the mean yield of the two sets of plots is significant.
   (*ii*) Whether the difference in the variability in yields is significant.

	Set of 40 plots	*Set of 60 plots*
Mean yield per plot	1258 lb	1243 lb
S.D. per plot	34	28

2. The yield of wheat in a random sample of 1000 farms in a certain area has a S.D. of 192 kg. Another random sample of 1000 farms gives a S.D. of 224 kg. Are the S.Ds significantly different ?

Answers

1. $z = 2.321$ Difference significant at 5% level; $z = 1.31$ Difference not significant at 5% level
2. $z = 4.851$ The S.Ds are significantly different.

6.14. TEST OF SIGNIFICANCE OF SMALL SAMPLES

When the size of the sample is less than 30, then the sample is called small sample. For such sample it will not be possible for us to assume that the random sampling distribution of a statistic is approximately normal and the values given by the sample data are sufficiently close to the population values and can be used in their place for the calculation of the standard error of the estimate.

t-TEST

6.15. STUDENT'S-t-DISTRIBUTION

(*P.T.U. 2007; M.D.U. Dec. 2007*)

This *t*-distribution is used when sample size is ≤ 30 and the population standard deviation is unknown.

t-statistic is defined as $t = \frac{\bar{x} - \mu}{s/\sqrt{n}} \sim t(n-1\,d.f.)$ d.f.-degrees of freedom where $s = \sqrt{\frac{\Sigma(X - \bar{X})^2}{n-1}}$.

The t-table

The *t*-table given at the end is the probability integral of *t*-distribution. The *t*-distribution has a different values for each degrees of freedom and when the degrees of freedom are infinitely large, the *t*-distribution is equivalent to normal distribution and the probabilities shown in the normal distribution tables are applicable.

Application of t-distribution

Some of the applications of *t*-distribution are given below:

1. *To test if the sample mean* ($\bar{X}$) *differs significantly from the hypothetical value* μ *of the population mean.*
2. *To test the significance between two sample means.*
3. *To test the significance of observed partial and multiple correlation coefficients.*

Critical value of t

The critical value or significant value of t at level of significance α degrees of freedom γ for two tailed test is given by

$$P[|t| > t_\gamma(\alpha)] = \alpha$$
$$P[|t| \leq t_\gamma(\alpha)] = 1 - \alpha$$

The significant value of t at level of significance α for a single tailed test can be got from those of two tailed test by referring to the values at 2α.

6.16. TEST I: t-TEST OF SIGNIFICANCE OF THE MEAN OF A RANDOM SAMPLE

To test whether the mean of a sample drawn from a normal population deviates significantly from a stated value when variance of the population is unknown.

H_0: There is no significant difference between the sample mean $\bar{x}$ and the population mean μ *i.e.*, we use the statistic

$$t = \frac{\overline{X} - \mu}{s/\sqrt{n}}, \qquad \text{where } \overline{X} \text{ is mean of the sample.}$$

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n} (X_i - \overline{X})^2 \text{ with degrees of freedom } (n-1).$$

At given level of significance α_1 and degrees of freedom $(n-1)$. We refer to t-table t_α (two tailed or one tailed).

If calculated t value is such that $|t| < t_\alpha$ the null hypothesis is accepted. $|t| > t_\alpha$ H_0 is rejected.

Fiducial limits of population mean

If t_α is the table of t at level of significance α at $(n-1)$ degrees of freedom.

$$\left|\frac{\overline{X} - \mu}{s/\sqrt{n}}\right| < t_\alpha \text{ for acceptance of } H_0.$$

$$\bar{x} - t_\alpha\, s/\sqrt{n} < \mu < \bar{x} + t_\alpha\, s/\sqrt{n}$$

95% confidence limits (level of significance 5%) are $\overline{X} \pm t_{0.05}\, s/\sqrt{n}$.

99% confidence limits (level of significance 1%) are $\overline{X} \pm t_{0.01}\, s/\sqrt{n}$.

Note. Instead of calculating s, we calculate S for the sample.

Since $s^2 = \dfrac{1}{n-1}\sum\limits_{i=1}^{n} (X_i - \overline{X})^2 \;\therefore\; S^2 = \dfrac{1}{n}\sum\limits_{i=1}^{n} (X_i - \overline{X})^2$. $\qquad \left[(n-1)s^2 = nS^2, s^2 = \dfrac{n}{n-1}S^2\right]$

ILLUSTRATIVE EXAMPLES

Example 1. *A random sample of size 16 has 53 as mean. The sum of squares of the derivation from mean is 135. Can this sample be regarded as taken from the population having 56 as mean? Obtain 95% and 99% confidence limits of the mean of the population.*

Sol. H_0: There is no significant difference between the sample mean and hypothetical population mean.

$$H_0: \mu = 56;\ H_1: \mu \neq 56 \quad \text{(Two tailed test)}$$

$$t: \frac{\overline{X} - \mu}{s/\sqrt{n}} \sim t(n - 1 \text{ d.f})$$

Given: $\overline{X} = 53, \mu = 56, n = 16, \Sigma(X - \overline{X})^2 = 135$

$$s = \sqrt{\frac{\Sigma(X - \overline{X})^2}{n-1}} = \sqrt{\frac{135}{15}} = 3;\ t = \frac{53 - 56}{3/\sqrt{16}} = \frac{-3 \times 4}{3} = -4$$

$$|t| = 4 .\ d.f\nu = 16 - 1 = 15.$$

Conclusion. $t_{0.05} = 1.753$.

Since $|t| = 4 > t_{0.05} = 1.753$ *i.e.*, the calculated value of t is more than the table value. The hypothesis is rejected. Hence the sample mean has not come from a population having 56 as mean.

95% confidence limits of the population mean

$$\overline{X} \pm \frac{s}{\sqrt{n}} t_{0.05},\ 53 \pm \frac{3}{\sqrt{16}} (1.725) = 51.706;\ 54.293$$

99% confidence limits of the population mean

$$\overline{X} \pm \frac{s}{\sqrt{n}} t_{0.01},\ 53 \pm \frac{3}{\sqrt{16}} (2.602) = 51.048;\ 54.951.$$

Example 2. *The life time of electric bulbs for a random sample of 10 from a large consignment gave the following data:*

Item	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Life in '000 hrs.	*4.2*	*4.6*	*3.9*	*4.1*	*5.2*	*3.8*	*3.9*	*4.3*	*4.4*	*5.6*

Can we accept the hypothesis that the average life time of bulb is 4000 hrs?

Sol. H_0: There is no significant difference in the sample mean and population mean. *i.e.*, $\mu = 4000$ hrs.

Applying the t-test: $t = \dfrac{\overline{X} - \mu}{s/\sqrt{n}} \sim t(10 - 1 d.f)$

X	4.2	4.6	3.9	4.1	5.2	3.8	3.9	4.3	4.4	5.6
$X - \overline{X}$	– 0.2	0.2	– 0.5	– 0.3	0.8	– 0.6	– 0.5	– 0.1	0	1.2
$(X - \overline{X})^2$	0.04	0.04	0.25	0.09	0.64	0.36	0.25	0.01	0	1.44

$$\overline{X} = \frac{\Sigma X}{n} = \frac{44}{10} = 4.4 \qquad \Sigma(X - \overline{X})^2 = 3.12$$

$$s = \sqrt{\frac{\Sigma(X - \overline{X})^2}{n-1}} = \sqrt{\frac{3.12}{9}} = 0.589;\ t = \frac{4.4 - 4}{\frac{0.589}{\sqrt{10}}} = 2.123$$

For $\gamma = 9$, $t_{0.05} = 2.26$.

Conclusion. Since the calculated value of t is less than table $t_{0.05}$. $\therefore$ The hypothesis $\mu = 4000$ hrs is accepted.

i.e., The average life time of bulbs could be 4000 hrs.

Example 3. *A sample of 20 items has mean 42 units and S.D. 5 units. Test the hypothesis that it is a random sample from a normal population with mean 45 units.*

Sol. H_0: There is no significant difference between the sample mean and the population mean

i.e., $\mu = 45$ units

$H_1: \mu \neq 45$ (Two tailed test)

Given: $n = 20, \overline{X} = 42, S = 5; \gamma = 19$ d.f.

$$s^2 = \frac{n}{n-1} S^2 = \left[\frac{20}{20-1}\right](5)^2 = 26.31 \quad \therefore \quad s = 5.129$$

Applying t-test $\quad t = \dfrac{\overline{X}-\mu}{s/\sqrt{n}} = \dfrac{42-45}{5.129/\sqrt{20}} = -2.615; \ |t| = 2.615$

The tabulated value of t at 5% level for 19 d.f. is $t_{0.05} = 2.09$.

Conclusion. Since $|t| > t_{0.05}$, the hypothesis H_0 is rejected. *i.e.,* there is significant difference between the sample mean and population mean.

i.e., The sample could not have come from this population.

Example 4. *The 9 items of a sample have the following values 45, 47, 50, 52, 48, 47, 49, 53, 51. Does the mean of these values differ significantly from the assumed mean 47.5.*

(M.D.U. Dec. 2011)

Sol. $H_0: \mu = 47.5$

i.e., there is no significant difference between the sample and population mean.

$H_1: \mu \neq 47.5$ (two tailed test); Given: $n = 9$, $\mu = 47.5$

X	45	47	50	52	48	47	49	53	51
$X-\overline{X}$	−4.1	−2.1	0.9	2.9	−1.1	−2.1	−0.1	3.9	1.9
$(X-\overline{X})^2$	16.81	4.41	0.81	8.41	1.21	4.41	0.01	15.21	3.61

$$\overline{X} = \frac{\Sigma x}{n} = \frac{442}{9} = 49.11; \ \Sigma(X-\overline{X})^2 = 54.89; \ s^2 = \frac{\Sigma(X-\overline{X})^2}{(n-1)} = 6.86 \quad \therefore \quad s = 2.619$$

Applying t-test $\quad t = \dfrac{\overline{X}-\mu}{s/\sqrt{n}} = \dfrac{49.1-47.5}{2.619/\sqrt{8}} = \dfrac{(1.6)\sqrt{8}}{2.619} = 1.7279$

$t_{0.05} = 2.31$ for $\gamma = 8$.

Conclusion. Since $|t| < t_{0.05}$, the hypothesis is accepted *i.e.,* there is no significant difference between their mean.

Example 5. *The following results are obtained from a sample of 10 boxes is biscuits.*

Mean weight content = 490 gm

S.D. of the weight = 9 gm could the sample come from a population having a mean of 500 gm?

Sol. Given: $\quad n = 10, \overline{X} = 490; S = 9$ gm, $\mu = 500$

$$s = \sqrt{\frac{n}{n-1}S^2} = \sqrt{\frac{10}{9}\times 9^2} = 9.486$$

H_0: The difference is not significant *i.e.,* $\mu = 500$; $H_1: \mu \neq 500$

Applying t-test $\quad t = \dfrac{\overline{X}-\mu}{s/\sqrt{n}} = \dfrac{490-500}{9.486/\sqrt{10}} = -0.333$

$t_{0.05} = 2.26$ for $\gamma = 9$.

Conclusion. Since $|t| = .333 > t_{0.05}$ the hypothesis H_0 is rejected *i.e.,* $\mu \neq 500$

$\therefore$ The sample could not have come from the population having mean 500 gm.

EXERCISE 6.6

1. Find the student's t for the following variable values in a sample of eight: −4, −2, −2, 0, 2, 2, 3, 3 taking the mean of the universe to be zero. *(P.T.U. 2007; M.D.U. May 2007, Dec. 2010)*
2. Ten individuals are choosen at random from a normal population of students and their marks found to be 63, 63, 66, 67, 68, 69, 70, 70, 71, 71. In the light of these data discuss the suggestion that mean mark of the population of students is 66.
3. The following values gives the lengths of 12 samples of egyptian cotton taken from a consignment: 48, 46, 49, 46, 52, 45, 43, 47, 47, 46, 45, 50. Test if the mean length of the consignment can be taken as 46.
4. A sample of 18 items has a mean 24 units and standard deviation 3 units. Test the hypothesis that it is a random sample from a normal population with mean 27 units.
5. A random sample of 10 boys had the I.Q's 70, 120, 110, 101, 88, 83, 95, 98, 107 and 100. Do these data support the assumption of a population mean I.Q of 160?
6. A filling machine is expected to fill 5 kg of powder into bags. A sample of 10 bags gave the following weights. 4.7, 4.9, 5.0, 5.1, 5.4, 5.2, 4.6, 5.1, 4.6 and 4.7. Test whether the machine is working properly. *(M.D.U. Dec. 2010)*
7. A machinist is making engine parts with axle-diameter of 0.7 unit. A random sample of 10 parts shows mean diameter 0.742 unit with a standard deviation of 0.04 unit. On the basis of this sample, what would you say that the work in inferior?

Answers

1. $t = 0.27$ **2.** accepted **3.** accepted
4. rejected **5.** accepted **6.** accepted
7. rejected.

6.17. TEST II: t-TEST FOR DIFFERENCE OF MEANS OF TWO SMALL SAMPLES (From a Normal Population)

This test is used to test whether the two samples $x_1, x_2, \ldots\ldots, x_{n_1}, y_1, y_2, \ldots\ldots, y_{n_2}$ of sizes n_1, n_2 have been drawn from two normal populations with mean μ_1 and μ_2 respectively under the assumption that the population variance are equal. $(\sigma_1 = \sigma_2 = \sigma)$.

H_0: The samples have been drawn from the normal population with means μ_1 and μ_2
i.e., $H_0: \mu_1 \neq \mu_2$.

Let $\overline{X}, \overline{Y}$ be their means of the two sample.

Under this H_0 the test of statistic t is given by $t = \dfrac{(\overline{X} - \overline{Y})}{s\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} \sim t(n_1 + n_2 - 2 \text{ d.f.})$

Note 1. If the two sample standard deviations s_1, s_2 are given then we have $s^2 = \dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}$.

Note 2. If $n_1 = n_2 = n$, $t = \dfrac{\overline{X} - \overline{Y}}{\sqrt{\dfrac{s_1^2 + s_2^2}{n-1}}}$ can be used as a test statistic.

Note 3. If the pairs of values are in some way associated (correlated) we can't use the test statistic as given in note 2. In this case we find the differences of the associated pairs of values and apply for single mean *i.e.*, $t = \dfrac{\overline{X} - \mu}{s/\sqrt{n}}$ with degrees of freedom $n - 1$.

The test statistic is $t = \dfrac{\overline{d}}{s/\sqrt{n}}$ or $t = \dfrac{\overline{d}}{s/\sqrt{n-1}}$, where $\overline{d}$ is the mean of paired difference

i.e., $d_i = x_i - y_i$

$\overline{d_i} = \overline{X} - \overline{Y}$, where (x_i, y_i) are the paired data $i = 1, 2, \ldots\ldots, n$.

ILLUSTRATIVE EXAMPLES

Example 1. *Two samples of sodium vapour bulbs were tested for length of life and the following results were got:*

	Size	*Sample mean*	*Sample S.D.*
Type I	*8*	*1234 hrs*	*36 hrs*
Type II	*7*	*1036 hrs*	*40 hrs*

Is the difference in the means significant to generalise that type I is superior to type II regarding length of life.

Sol. $H_0: \mu_1 = \mu_2$ *i.e.*, two types of bulbs have same lifetime.

$H_1: \mu_1 > \mu_2$ *i.e.*, type I is superior to Type II

$$s^2 = \frac{n_1 s_2^2 + n_2 s_2^2}{n_1 + n_2 - 2} = \frac{8 \times (36)^2 + 7(40)^2}{8 + 7 - 2} = 1659.076 \quad \therefore \quad s = 40.7317$$

The t-statistic $\quad t = \dfrac{\overline{X}_1 - \overline{X}_2}{s\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} = \dfrac{1234 - 1036}{40.7317\sqrt{\dfrac{1}{8} + \dfrac{1}{7}}} = 18.1480 \sim t(n_1 + n_2 - 2 \text{ d.f.})$

$t_{0.05}$ at d.f. 13 is 1.77 (One tailed test)

Conclusion. Since calculated $|t| > t_{0.05}$ H_0 is rejected *i.e.*, H_1 is accepted.

$\therefore$ Type I is definitely superior to type II.

where $\overline{X} = \sum_{i=1}^{n_1} \dfrac{X_i}{n_i}$, $\quad \overline{Y} = \sum_{j=1}^{n_2} \dfrac{Y_j}{n_2}$; $\quad s^2 = \dfrac{1}{n_1 + n_2 - 2} [\Sigma(X_i - \overline{X})^2 + (Y_j - \overline{Y})^2]$

is an unbiased estimate of the population variance σ^2.

t follows t distribution with $n_1 + n_2 - 2$ degrees of freedom.

Example 2. *Samples of sizes 10 and 14 were taken from two normal populations with S.D. 3.5 and 5.2. The sample means were found to be 20.3 and 18.6. Test whether the means of the two populations are the same at 5% level.*

Sol. $H_0: \mu_1 = \mu_2$ *i.e.*, the means of the two populations are the same.

$H_1: \mu_1 \neq \mu_2$.

Given $\overline{X}_1 = 20.3$, $\overline{X}_2 = 18.6$; $n_1 = 10$, $n_2 = 14$, $s_1 = 3.5$, $s_2 = 5.2$

$$s^2 = \frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2} = \frac{10(3.5)^2 + 14(5.2)^2}{10 + 14 - 2} = 22.775 \quad \therefore \quad s = 4.772$$

$$t = \frac{\overline{X}_1 - \overline{X}_2}{s\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} = \frac{20.3 - 18.6}{\left(\sqrt{\dfrac{1}{10} + \dfrac{1}{14}}\right) 4.772} = 0.8604$$

The value of t at 5% level for 22 d.f. is $t_{0.05} = 2.0739$.

Conclusion. Since $|t| = 0.8604 < t_{0.05}$ the hypothesis is accepted. *i.e.*, there is no significant difference between their means.

Example 3. *The height of 6 randomly chosen sailors are in inches are 63, 65, 68, 69, 71 and 72. Those of 9 randomly chosen soldiers are 61, 62, 65, 66, 69, 70, 71, 72 and 73. Test whether the sailors are on the average taller than soldiers.*

Sol. Let X_1 and X_2 be the two samples denoting the heights of sailors and soldiers.

Given the sample size $n_1 = 6$, $n_2 = 9$, $H_0: \mu_1 = \mu_2$.

i.e., the mean of both the population are the same.

$H_1: \mu_1 > \mu_2$ (one tailed test)

Calculation of two sample mean:

X_1	63	65	68	69	71	72
$X_1 - \overline{X}_1$	– 5	– 3	0	1	3	4
$(X_1 - \overline{X}_1)^2$	25	9	0	1	9	16

$$\overline{X}_1 = \frac{\Sigma X_1}{n_1} = 68; \quad \Sigma(X_1 - \overline{X}_1)^2 = 60$$

X_2	61	62	65	66	69	70	71	72	73
$X_2 - \overline{X}_2$	– 6.66	– 5.66	– 2.66	1.66	1.34	2.34	3.34	4.34	5.34
$(X_2 - \overline{X}_2)^2$	44.36	32.035	7.0756	2.7556	1.7956	5.4756	11.1556	18.8356	28.5156

$$\overline{X}_2 = \frac{\Sigma X_2}{n_2} = 67.66;\quad \Sigma(X_2 - \overline{X}_2)^2 = 152.0002$$

$$s^2 = \frac{1}{n_1 + n_2 - 2}\,[\Sigma(X_1 - \overline{X}_1)^2 + \Sigma(X_2 - \overline{X}_2)^2]$$

$$= \frac{1}{6 + 9 - 2}\,[60 + 152.0002] = 16.3077 \;\Rightarrow\; s = 4.038$$

Under H_0,
$$t = \frac{\overline{X}_1 - \overline{X}_2}{s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{68 - 67.666}{4.0382\sqrt{\frac{1}{6} + \frac{1}{9}}} = 0.3031 \sim t(n_1 + n_2 - 2 \text{ d.f.})$$

The value of t at 10% level of significance ($\because$ the test is one tailed) for 13 d.f. is 1.77.

Conclusion. Since $|t| = 0.3031 < t_{0.05} = 1.77$ the hypothesis H_0 is accepted.

i.e., there is no significance difference between their average.

i.e., the sailors are not on the average taller than the soldiers.

Example 4. *A certain stimulus administered to each of 12 patients resulted in the following increases of blood pressure: 5, 2, 8, – 1, 3, 0, – 2, 1, 5, 0, 4, 6. Can it be concluded that the stimulus will in general be accompanied by an increase in blood pressure?*

(V.T.U. 2007; M.D.U. Dec. 2011)

Sol. To test whether the mean increase in blood pressure of all patients to whom the stimulus is administered will be positive. We have to assume that this population is normal with mean μ and S.D. σ which are unknown.

$$H_0: \mu = 0;\; H_1: \mu_1 > 0$$

The test statistic under H_0

$$t = \frac{\overline{d}}{s/\sqrt{n-1}} \sim t(n - 1 \text{ degrees of freedom})$$

$$\overline{d} = \frac{5 + 2 + 8 + (-1) + 3 + 0 + 6 + (-2) + 1 + 5 + 0 + 4}{12} = 2.583$$

$$s^2 = \frac{\Sigma d^2}{n} - \overline{d}^2 = \frac{1}{12}\,[5^2 + 2^2 + 8^2 + (-1)^2 + 3^2 + 0^2 + 6^2 + (-2)^2 + 1^2 + 5^2 + 0^2 + 4^2] - (2.583)^2$$

$$= 8.744 \quad \therefore \quad s = 2.9571$$

$$t = \frac{\overline{d}}{s/\sqrt{n-1}} = \frac{2.583}{2.9571/\sqrt{12-1}} = \frac{2.583\sqrt{11}}{2.9571} = 2.897 \sim t(n - 1 \text{ d.f.})$$

Conclusion. The tabulated value of $t_{0.05}$ at 11 d.f. is 2.2.

$\because$ $|t| > t_{0.05}$, H_0 is rejected.

i.e., the stimulus does not increase the blood pressure. The stimulus in general will be accompanied by increase in blood pressure.

Example 5. *Memory capacity of 9 students was tested before and after a course of meditation for a month. State whether the course was effective or not from the data below (in same units)*

Before	*10*	*15*	*9*	*3*	*7*	*12*	*16*	*17*	*4*
After	*12*	*17*	*8*	*5*	*6*	*11*	*18*	*20*	*3*

Sol. Since the data are correlated and concerned with the same set of students we use paired t-test.

H_0: Training was not effective $\mu_1 = \mu_2$

H_1: $\mu_1 \neq \mu_2$ (Two tailed test).

Before training (X)	*After training (Y)*	$d = X - Y$	d^2
10	12	– 2	4
15	17	– 2	4
9	8	1	1
3	5	– 2	4
7	6	1	1
12	11	1	1
16	18	– 2	4
17	20	– 3	9
4	3	1	1
		$\Sigma d = -7$	$\Sigma d^2 = 29$

$$\overline{d} = \frac{\Sigma d}{n} = \frac{-7}{9} = -0.7778;\; s^2 = \frac{\Sigma d^2}{n} - (\overline{d})^2 = \frac{29}{9} - (-0.7778)^2 = 2.617$$

$$t = \frac{\overline{d}}{s/\sqrt{n-1}} = \frac{-0.7778}{\sqrt{2.6172}/\sqrt{8}} = \frac{-0.7778 \times \sqrt{8}}{1.6177} = -1.359$$

The tabulated value of $t_{0.05}$ at 8 d.f. is 2.31.

Conclusion. Since $|t| = 1.359 < t_{0.05}$, H_0 is accepted *i.e.*, training was not effective in improving performance.

Example 6. *The following figures refer to observations in live independent samples.*

Sample I	*25*	*30*	*28*	*34*	*24*	*20*	*13*	*32*	*22*	*38*
Sample II	*40*	*34*	*22*	*20*	*31*	*40*	*30*	*23*	*36*	*17*

Analyse whether the samples have been drawn from the populations of equal means.

Sol. H_0: The two samples have been drawn from the population of equal means. *i.e.*, there is no significant difference between their means

i.e., $\mu_1 = \mu_2$

H_1: $\mu_1 \neq \mu_2$ (Two tailed test)

Given n_1 = Sample I size = 10; n_2 = Sample II size = 10

To calculate the two sample mean and sum of squares of deviation from mean. Let X_1 be the sample I and X_2 be the sample II.

X_1	25	30	28	34	24	20	13	32	22	38
$X_1 - \bar{X}_1$	− 1.6	3.4	1.4	7.4	− 2.6	− 6.6	− 13.6	5.4	4.6	11.4
$(X_1 - \bar{X}_1)^2$	2.56	11.56	1.96	54.76	6.76	43.56	184.96	29.16	21.16	129.96
X_2	40	34	22	20	31	40	30	23	36	17
$X_2 - \bar{X}_2$	10.7	4.7	− 7.3	− 9.3	1.7	10.7	0.7	− 6.3	6.7	− 12.3
$(X_2 - \bar{X}_2)^2$	114.49	22.09	53.29	86.49	2.89	114.49	0.49	39.67	44.89	151.29

$$\bar{X}_1 = \sum_{i=1}^{10} \frac{X_1}{n_1} = 26.6 \qquad \bar{X}_2 = \sum_{i=1}^{10} \frac{X_2}{n_2} = \frac{293}{10} = 29.3$$

$$\Sigma(X_1 - \bar{X}_1)^2 = 486.4 \qquad \Sigma(X_2 - \bar{X}_2)^2 = 630.08$$

$$s^2 = \frac{1}{n_1 + n_2 - 2}[\Sigma(X_1 - \bar{X}_1)^2 + \Sigma(X_2 - \bar{X}_2)^2]$$

$$= \frac{1}{10 + 10 - 2}[486.4 + 630.08] = 62.026 \qquad \therefore \; s = 7.875$$

Under H_0 the test statistic is given by

$$t = \frac{\bar{X}_1 - \bar{X}_2}{s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{26.6 - 29.3}{7.875\sqrt{\frac{1}{10} + \frac{1}{10}}} = -0.7666 \sim t(n_1 + n_2 - 2 \text{ d.f.})$$

$$|\,t\,| = 0.7666.$$

Conclusion. The tabulated value of t at 5% level of significance for 18 d.f. is 2.1. Since the calculated value $|\,t\,| = 0.7666 < t_{0.05}$, H_0 is accepted.

i.e., there is no significant difference between their means.

i.e., the two samples have been drawn from the populations of equal means.

EXERCISE 6.7

1. The mean life of 10 electric motors was found to be 1450 hrs with S.D. of 423 hrs. A second sample of 17 motors chosen from a different batch showed a mean life of 1280 hrs with a S.D. of 398 hrs. Is there a significant difference between means of the two samples.
2. The marks obtained by a group of 9 regular course students and another group of 11 part time course students in a test are given below

Regular	56	62	63	54	60	51	67	69	58		
Part-time	62	70	71	62	60	56	75	64	72	68	66

Examine whether the marks obtained by regular students and part-time students differ significantly at 5% and 1% level of significance.

3. A group of 10 rats fed on diet A and another group of 8 rats fed on a different diet B recorded the following increase in weight (gm).

Diet A	5	6	8	1	12	4	3	9	6	10
Diet B	2	3	6	8	10	1	2	8		

Does it show the superiority of diet A over the diet B? (*M.D.U. May 2011*)

4. Two independent samples of sizes 7 and 9 have the following values:

Sample A	10	12	10	13	14	11	10		
Sample B	10	13	15	12	10	14	11	12	11

Test whether the difference between the mean is significant. (*P.T.U. B.Pharma, May 2005*)

5. To compare the prices of a certain product in two cities, 10 shops when related at random in each town. The price was noted below:

City 1	61	63	56	63	56	63	59	56	44	61
City 2	55	54	47	59	51	61	57	54	64	58

Test whether the average prices can be said to be the same in two cities.

6. The average number of articles produced by two machines per day are 200 and 250 with standard deviation 20 and 25 respectively on the basis of records of 25 days production. Can you regard both the machines equally efficient at 5% level of significance?
7. Two salesman represent a firm in a certain company. One of them claims that he makes large sales than the other. A sample survey was made and the following results were obtained:

No. of sales	1st Salesman (18)	2nd Salesman (20)
Average sales	₹ 210	₹ 175
S.D.	₹ 25	₹ 20

Find if the average sales differ significantly.

8. Two types of drugs A and B were used on 5 and 7 patients respectively for reducing their weights. The decrease in the weight after using drugs for six months are as follows:

Drug A	10	12	13	11	14		
Drug B	8	9	12	14	15	10	9

Is there a significant difference in the efficiency of the two drugs? If not, which drug you should buy? (*P.T.U. Dec. 2005*)

9. Nine patients, to whom a certain drug was administered, registered the following rise in blood pressure:

   3, 7, 4 − 1, − 3, 6, − 4, 1, 5

   Test the hypothesis that the drug does not raise the blood pressure.
10. The change in sleeping hours of 7 patients after taking a medicine are as follows:

    0.7, 0.1, − 0.3, 1.2, 1.0, 0.3, − 0.4 hrs.

    Do these data give evidence that the medicine produces additional hours of sleep?

Answers

1. accepted	**2.** rejected	**3.** accepted	**4.** accepted
5. accepted	**6.** rejected	**7.** rejected	**8.** accepted
9. rejected	**10.** rejected.		

(P.T.U. 2005, 2006

6.18. CHI-SQUARE (χ^2) TEST

When a coin is tossed 200 times, the theoretical considerations lead us to expec 100 heads and 100 tails. But in practice, these results are rarely achieved. The quantity χ^2 (Greek letter, pronounced as chi-square) describes the magnitude of discrepancy between theor and observation. If $\chi = 0$, the observed and expected frequencies completely coincide. Th greater the discrepancy between the observed and expected frequencies, the greater is th value of χ^2. Thus χ^2 **affords a measure of the correspondence between theory an observation.**

If O_i $(i = 1, 2, ..., n)$ is a set of observed (experimental) frequencies and E_i $(i = 1, 2, ..., n$ is the corresponding set of expected (theoretical or hypothetical) frequencies, then, χ^2 i **defined as**

$$\chi^2 = \sum_{i=1}^{n} \left[\frac{(O_i - E_i)^2}{E_i} \right]$$

where $\Sigma O_i = \Sigma E_i = N$ (total frequency) and degrees of freedom $(d.f.) = (n - 1)$.

Note. (*i*) If $\chi^2 = 0$, the observed and theoretical frequencies agree exactly.

(*ii*) If χ^2 0 they do not agree exactly.

6.19. DEGREE OF FREEDOM

While comparing the calculated value of χ^2 with the table value, we have to determin the degrees of freedom.

If we have to choose any four numbers whose sum is 50, we can exercise our independ ent choice for any three numbers only, the fourth being 50 minus the total of the three num bers selected. Thus, though we were to choose any four numbers, our choice was reduced t three because of one condition imposed. There was only one restraint on our freedom and ou degrees of freedom were $4 - 1 = 3$. If two restrictions are imposed, our freedom to choose will b further curtailed and degrees of freedom will be $4 - 2 = 2$.

In general, the number of degrees of freedom is the total number of observations les the number of independent constraints imposed on the observations. Degrees of freedom (*d.f* are usually denoted by ν (the letter 'nu' of the Greek alphabet).

Thus, $\nu = n - k$, where k is the number of independent constraints in a set of data of observations.

Note 1. For a $p \times q$ contingency table (p columns and q rows).

$$\nu = (p - 1)(q - 1)$$

2. In the case of a contingency table, the expected frequency of any class

$$= \frac{\text{Total of row in which it occurs} \times \text{Total of col. in which it occurs}}{\text{Total number of observations}}$$

3. χ^2 test is one of the simplest and the most general test known. It is applicable to a very large number of problems in practice which can be summed up under the following heads :

(*i*) as a test of goodness of fit. (*ii*) as a test of independence of attributes.

(*iii*) as a test of homogeneity of independent estimates of the population variance.

(*iv*) as a test if the hypothetical value of the population variance σ^2.

(*v*) as a list to the homogeneity of independent estimates of the population correlation coefficient.

6.20. CONDITIONS FOR APPLYING χ^2 TEST

Following are the conditions which should be satisfied before χ^2 test can be applied.

(*a*) N, the total number of frequencies should be large. It is difficult to say what constitutes largeness, but as an arbitrary figure, we may say that **N should be at least 50,** however, few the cells.

(*b*) No theoretical cell-frequency should be small. Here again, it is difficult to say what constitutes smallness, but 5 should be regarded as the very minimum and **10 is better**. If small theoretical frequencies occur (*i.e.*, < 10), the difficulty is overcome by grouping two or more classes together before calculating (O – E). **It is important to remember that the number of degrees of freedom is determined with the number of classes after regrouping.**

(*c*) The constraints on the cell frequencies, if any, should be linear.

Note. If any one of the theoretical frequency is less than 5, then we apply a corrected given by F Yates, which is usually known as 'Yates correction for continuity', we add 0.5 to the cell frequency which is less than 5 and adjust the remaining cell frequency suitably so that the marginal total is not changed.

6.21. THE χ^2 DISTRIBUTION

For large sample sizes, the sampling distribution of χ^2 can be closely approximated by a continuous curve known as the chi-square distribution. The probability function of χ^2 distribution is given by

$$f(\chi)^2 = c(\chi^2)^{(\nu/2 - 1)} \ e^{-x^2/2}$$

where $e = 2.71828$, ν = number of degrees of freedom; c = a constant depending only on ν.

Symbolically, the degrees of freedom are denoted by the symbol ν or by d.f. and are obtained by the rule $\nu = n - k$, where k refers to the number of independent constraints.

In general, when we fit a binomial distribution the number of degrees of freedom is one less than the number of classes; when we fit a Poisson distribution the degrees of freedom are 2 less than the number of classes, because we use the total frequency and the arithmetic mean to get the parameter of the Poisson distribution. When we fit a normal curve the number of degrees of freedom are 3 less than the number of classes, because in this fitting we use the total frequency, mean and standard deviation.

If the data is given in a series of "n" number then degrees of freedom = $n - 1$.

In the case of Binomial distribution d.f. = $n - 1$

In the case of Poisson distribution d.f. = $n - 2$

In the case of Normal distribution d.f. = $n - 3$.

6.22. χ^2 TEST AS A TEST OF GOODNESS OF FIT

(M.D.U. Dec. 2007)

χ^2 test enables us to ascertain how well the theoretical distributions such as Binomial, Poisson or Normal etc. fit empirical distributions, *i.e.*, distributions obtained from sample data. If the **calculated value of χ^2 is less than the table value** at a specified level (generally 5%) of significance, the **fit is considered to be good** *i.e.*, the divergence between actual and expected frequencies is attributed to fluctuations of simple sampling. If the calculated value of χ^2 is greater than the table value, the fit is considered to be poor.

ILLUSTRATIVE EXAMPLES

Example 1. *The following table gives the number of accidents that took place in an industry during various days of the week. Test if accidents are uniformly distributed over the week.*

Day	*Mon*	*Tue*	*Wed*	*Thu*	*Fri*	*Sat*
No. of accidents	*14*	*18*	*12*	*11*	*15*	*14*

Sol. H_0: Null hypothesis. The accidents are uniformly distributed over the week.

Under this H_0, the expected frequencies of the accidents on each of these days $= \dfrac{84}{6} = 14$

Observed frequency O_i	14	18	12	11	15	14
Expected frequency E_i	14	14	14	14	14	14
$(O_i - E_i)^2$	0	16	4	9	1	0

$$\chi^2 = \frac{\Sigma(O_i - E_i)^2}{E_i} = \frac{30}{14} = 2.1428.$$

Conclusion. Table value of χ^2 at 5% level for $(6 - 1 = 5$ d.f.) is 11.09.

Since the calculated value of χ^2 is less than the tabulated value H_0 is accepted *i.e.*, the accidents are uniformly distributed over the week.

Example 2. *A die is thrown 270 times and the results of these throws are given below:*

No. appeared on the die	*1*	*2*	*3*	*4*	*5*	*6*
Frequency	*40*	*32*	*29*	*59*	*57*	*59*

Test whether the die is biased or not.

Sol. Null hypothesis H_0: Die is unbiased.

Under this H_0, the expected frequencies for each digit is $\dfrac{276}{6} = 46$

To find the value of χ^2

O_i	40	32	29	59	57	59
E_i	46	46	46	46	46	46
$(O_i - E_i)^2$	36	196	289	169	121	169

$$\chi^2 = \frac{\Sigma(O_i - E_i)^2}{E_i} = \frac{980}{46} = 21.30.$$

Conclusion. Tabulated value of χ^2 at 5% level of significance for $(6 - 1 = 5)$ d.f. is 11.09. Since the calculated value of $\chi^2 = 21.30 > 11.07$ the tabulated value, H_0 is rejected.

i.e., Die is not unbiased or Die is biased.

Example 3. *The following table shows the distribution of digits in numbers chosen at random from a telephone directory.*

Digits	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
Frequency	*1026*	*1107*	*997*	*966*	*1075*	*933*	*1107*	*972*	*964*	*853*

Test whether the digits may be taken to occur equally frequently in the directory.

Sol. Null hypothesis H_0: The digits taken in the directory occur equally frequency. *i.e.*, there is no significant difference between the observed and expected frequency.

Under H_0, the expected frequency is given by $= \dfrac{10{,}000}{10} = 1000$

To find the value of χ^2

O_i	1026	1107	997	996	1075	1107	933	972	964	853
E_i	1000	1000	1000	1000	1000	1000	1107	1000	1000	1000
$(O_i - E_i)^2$	676	11449	9	1156	5625	11449	4489	784	1296	21609

$$\chi^2 = \frac{\Sigma(O_i - E_i)^2}{E_i} = \frac{58542}{1000} = 58.542.$$

Conclusion. The tabulated value of χ^2 at 5% level of significance for 9 d.f. is 16.919. Since the calculated value of χ^2 is greater than the tabulated value, H_0 is rejected.

i.e., there is significant difference between the observed and theoretical frequency.

i.e., the digits taken in the directory do not occur equally frequently.

Example 4. *Records taken of the number of male and female births in 800 families having four children are as follows:*

No. of male births	*0*	*1*	*2*	*3*	*4*
No. of female births	*4*	*3*	*2*	*1*	*0*
No. of families	*32*	*178*	*290*	*236*	*94*

Test whether the data are consistent with the hypothesis that the binomial law holds and the chance of male birth is equal to that of female birth, namely $p = q = 1/2$.

(P.T.U. 2006; U.P.T.U. 2009)

Sol. H_0: The data are consistent with the hypothesis of equal probability for male and female births. *i.e.*, $p = q = 1/2$.

We use Binomial distribution to calculate theoretical frequency given by:

$N(r) = N \times P(X = r)$

where N is the total frequency. $N(r)$ is the number of families with r male children:

$P(X = r) = {}^nC_r p^r q^{n-r}$

where p and q are probability of male and female birth, n is the number of children.

$$N(0) = \text{No. of families with 0 male children} = 800 \times {}^4C_0 \left(\frac{1}{2}\right)^4 = 800 \times 1 \times \frac{1}{2^4} = 50$$

$$N(1) = 800 \times {}^4C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^3 = 200; \; N(2) = 800 \times {}^4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2 = 300$$

$$N(3) = 800 \times {}^4C_3 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^3 = 200; \; N(4) = 800 \times {}^4C_4 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^4 = 50$$

Observed frequency O_i	32	178	290	236	94
Exp-frequency E_i	50	200	300	200	50
$(O_i - E_i)^2$	324	484	100	1296	1936
$\frac{(O_i - E_i)^2}{E_i}$	6.48	2.42	0.333	6.48	38.72

$$\chi^2 = \frac{\Sigma(O_i - E_i)^2}{E_i} = 54.433.$$

Conclusion. Table value of χ^2 at 5% level of significance for $5 - 1 = 4$ d.f. is 9.49.

Since the calculated value of χ^2 is greater than the tabulated value, H_0 is rejected.

i.e., the data are not consistent with the hypothesis that the Binomial law holds and that the chance of a male birth is not equal to that of a female birth.

Note. Since the fitting is binomial, the degrees of freedom $\nu = n - 1$ *i.e.*, $\nu = 5 - 1 = 4$.

Example 5. *Verify whether Poisson distribution can be assumed from the data given below:*

No. of defects	*0*	*1*	*2*	*3*	*4*	*5*
Frequency	*6*	*13*	*13*	*8*	*4*	*3*

(P.T.U. 2007)

Sol. H_0: Poisson fit is a good fit to the data.

$$\text{Mean of the given distribution} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{94}{97} = 2$$

To fit a Poisson distribution we require m. Parameter $m = \bar{x} = 2$.

By Poisson distribution the frequency of r success is

$$N(r) = N \times e^{-m} \cdot \frac{m^r}{r!}, \text{ N is the total frequency.}$$

$$N(0) = 47 \times e^{-2} \cdot \frac{(2)^0}{0!} = 6.36 \approx 6; \; N(1) = 47 \times e^{-2} \cdot \frac{(2)^1}{1!} = 12.72 = 13$$

$$N(2) = 47 \times e^{-2} \cdot \frac{(2)^2}{2!} = 12.72 = 13; \; N(3) = 47 \times e^{-2} \cdot \frac{(2)^3}{3!} = 8.48 \approx 9$$

$$N(4) = 47 \times e^{-2} \cdot \frac{(2)^4}{4!} = 4.24 \approx 4; \; N(5) = 47 \times e^{-2} \cdot \frac{(2)^5}{5!} = 1.696 = 2.$$

X	0	1	2	3	4	5
O_i	6	13	13	8	4	3
E_i	6.36	12.72	12.72	8.48	4.24	1.696
$\frac{(O_i - E_i)^2}{E_i}$	0.2037	0.00616	0.00616	0.02716	0.0135	1.0026

$$\chi^2 = \frac{\Sigma(O_i - E_i)^2}{E_i} = 1.2864.$$

Conclusion. The calculated value of χ^2 is 1.2864. Tabulated value of χ^2 at 5% level of significance for $\gamma = 6 - 2 = 4$ d.f. is 9.49. Since the calculated value of χ^2 is less than that of tabulated value. H_0 is accepted *i.e.*, Poisson distribution provides a good fit to the data.

Example 6. *The theory predicts the proportion of beans in the four groups, G_1, G_2, G_3, G_4 should be in the ratio 9: 3: 3: 1. In an experiment with 1600 beans the numbers in the four groups were 882, 313, 287 and 118. Does the experimental result support the theory?*

Sol. H_0. The experimental result support the theory. *i.e.*, there is no significant difference between the observed and theoretical frequency under H_0, the theoretical frequency can be calculated as follows:

$$E(G_1) = \frac{1600 \times 9}{16} = 900; \; E(G_2) = \frac{1600 \times 3}{16} = 300;$$

$$E(G_3) = \frac{1600 \times 3}{16} = 300; \; E(G_4) = \frac{1600 \times 1}{16} = 100$$

To calculate the value of χ^2.

Observed frequency O_i	882	313	287	118
Exp. frequency E_i	900	300	300	100
$\frac{(O_i - E_i)^2}{E_i}$	0.36	0.5633	0.5633	3.24

$$\chi^2 = \frac{\Sigma(O_i - E_i)^2}{E_i} = 4.7266.$$

Conclusion. Table value of χ^2 at 5% level of significance for 3 d.f. is 7.815. Since the calculated value of χ^2 is less than that of the tabulated value. Hence H_0 is accepted. *i.e.*, the experimental result support the theory.

EXERCISE 6.8

1. The following table gives the frequency of occupance of the digits 0, 1, ..., 9 in the last place in four logarithm of numbers 10-99. Examine if there is any peculiarity.

Digits	0	1	2	3	4	5	6	7	8	9
Frequency	6	16	15	10	12	12	3	2	9	5

2. The sales in a supermarket during a week are given below. Test the hypothesis that the sales do not depend on the day of the week, using a significant level of 0.05.

Days	Mon	Tues	Wed	Thus	Fri	Sat
Sales (in 1000 ₹)	65	54	60	56	71	84

3. A survey of 320 families with 5 children each revealed the following information:

No. of boys	5	4	3	2	1	0
No. of girls	0	1	2	3	4	5
No. of families	14	56	110	88	40	12

Is this result consistent with the hypothesis that male and female birth are equally probable?

4. 4 coins were tossed at a time and this operation is repeated 160 times. It is found that 4 heads occur 6 times, 3 heads occur 43 times, 2 heads occur 69 times, one head occur 34 times. Discuss whether the coin may be regarded as unbiased?

5. Fit a Poisson distribution to the following data and test the goodness of fit.

x	0	1	2	3	4
f	109	65	22	3	1

6. Fit a Poisson distribution to the following data and test the goodness of fit.

x	0	1	2	3	4
f	46	38	22	9	1

(*M.D.U. Dec. 2010*)

7. (*a*) Fit a binomial distribution to the following data and test the goodness of fit.

x	0	1	2	3	4	5	6
f	13	25	52	58	32	16	4

(*b*) A set of five similar coins is tossed 320 times and the result is

No. of heads	0	1	2	3	4	5
Frequency	6	27	72	112	71	32

Test the hypothesis that the data follow a Binomial distribution. (*M.D.U. Dec. 2010*)

8. 200 digits are chosen at random from a set of tables. The frequencies of the digits are as follows:

Digit	0	1	2	3	4	5	6	7	8	9
Frequency	18	19	23	21	16	25	22	20	21	15

Use chi-square test to assess the correctness of the hypothesis that the digits were distributed in equal numbers in the tables from where they are chosen.

9. In the accounting department of bank, 100 accounts are selected at random and estimated for errors. The following results were obtained:

No. of errors	0	1	2	3	4	5	6
No. of accounts	35	40	19	2	0	2	2

Does this information verify that the errors are distributed according to the Poisson probability law?

10. A sample analysis of examination results of 500 students, it was found that 280 students have failed, 170 have secured a third class, 90 have secured a second class and the rest, a first class. Do these figures support the general belief that above categories are in the ratio 4 : 3 : 2 : 1 respectively?

Answers

1. no **2.** accepted **3.** accepted **4.** unbiased

5. Poisson law fits the data **6.** Poisson law fits the data

7. (*a*) Binomial law does not fit the data (*b*) Rejected **8.** accepted **9.** may be

10. yes.

6.23. χ^2-TEST AS A TEST OF INDEPENDENCE

With the help of χ^2 test, we can find whether or not two attributes are associated. We take the null hypothesis that there is no association between the attributes under study, *i.e.*, **we assume that the two attributes are independent. If the calculated value of χ^2 is less than the table value** at a specified level (generally 5%) of significance, the hypothesis holds good, *i.e.*, **the attributes are independent** and do not bear any association. On the other hand, if the calculated value of χ^2 is greater than the table value at a specified level of significance, we say that the results of the experiment do not support the hypothesis. In other words, the attributes are associated. Thus a very useful application of χ^2 test is to investigate the relationship between trials or attributes which can be classified into two or more categories.

The sample data set out into two-way table, called **contingency table.**

Let us consider two attributes A and B divided into r classes $A_1, A_2, A_3, \ldots\ldots, A_r$ and B divided into s classes $B_1, B_2, B_3, \ldots\ldots, B_s$. If $(A_i), (B_j)$ represents the number of persons possessing the attributes A_i, B_j respectively, $(i = 1, 2, \ldots\ldots, r, j = 1, 2, \ldots\ldots, s)$ and $(A_i B_j)$ represent the number of persons possessing attributes A_i and B_j. Also we have $\sum_{i=1}^{r} A_i = \sum_{j=1}^{s} B_j = N$ where N is the total frequency. The contingency table for $r \times s$ is given below:

B \ A	A_1	A_2	A_3	$..A_r$	*Total*
	(A_1B_1)	(A_2B_1)	(A_3B_1)	...... (A_rB_1)	B_1
B_1	(A_1B_1)	(A_2B_1)	(A_3B_1)	...... (A_rB_1)	B_1
B_2	(A_1B_2)	(A_2B_2)	(A_3B_2)	...... (A_rB_2)	B_2
B_3	(A_1B_3)	(A_2B_3)	(A_3B_3)	...... (A_rB_3)	B_3
......	......	......	......	......	......
......	......	......	......	......	......
B_s	(A_1B_s)	(A_2B_s)	(A_3B_s)	...... (A_rB_s)	(B_s)
Total	(A_1)	(A_2)	(A_3)	...... (A_r)	N

H_0: Both the attributes are independent. *i.e.*, A and B are independent under the null hypothesis, we calculate the expected frequency as follows:

$$P(A_i) = \text{Probability that a person possesses the attribute } A_i = \frac{(A_i)}{N},\ i = 1, 2, \ldots\ r$$

$$P(B_j) = \text{Probability that a person possesses the attribute } B_j = \frac{(B_j)}{N}$$

$$P(A_iB_j) = \text{Probability that a person possesses both attributes } A_i \text{ and } B_j = \frac{(A_iB_j)}{N}$$

If $(A_iB_j)_0$ is the expected number of persons possessing both the attributes A_i and B_j

$$(A_iB_j)_0 = NP(A_iB_j) = NP(A_i)(B_j)$$

$$= N\frac{(A_i)}{N}\frac{(B_j)}{N} = \frac{(A_iB_j)}{N} \qquad (\because \text{ A and B are independent})$$

Hence
$$\chi^2 = \sum_{i=1}^{r}\sum_{j=1}^{s}\left[\frac{[(A_iB_j) - (A_iB_j)_0]^2}{(A_iB_j)_0}\right]$$

which is distributed as a χ^2 variate with $(r-1)(s-1)$ degrees of freedom.

Note 1. For a 2×2 contingency table where the frequencies are $\frac{a}{b}\Big|\frac{b}{d}$ χ^2 can be calculated from independent frequencies as $\chi^2 = \dfrac{(a+b+c+d)(ad-bc)^2}{(a+b)(c+d)(b+d)(a+c)}$.

Note 2. If the contingency table is not 2×2, then the formula for calculating χ^2 as given in note 1, can't be used. Hence we have another formula for calculating the expected frequency $(A_iB_j)_0 = \dfrac{(A_i)(B_j)}{N}$

i.e., expected frequency in each cell is $= \dfrac{\text{Product of column total and row total}}{\text{whole total}}$.

Note 3. If $\frac{a}{c}\Big|\frac{b}{d}$ is the 2×2 contingency table with two attributes, $Q = \dfrac{ad - bc}{ad + bc}$ is called the coefficient of association.

If the attributes are independent then $\dfrac{a}{b} = \dfrac{c}{d}$.

Note 4. Yates's Correction. In a 2×2 table, if the frequencies of a cell is small, we make Yate's correction to make χ^2 continuous.

Decrease by $\frac{1}{2}$ those cell frequencies which are greater than expected frequencies, and increase by $\frac{1}{2}$ those which are less than expectation. This will not affect the marginal columns. This correction is known as Yate's correction to continuity.

After Yate's correction
$$\chi^2 = \frac{N\left(bc - ad - \frac{1}{2}N\right)^2}{(a+c)(b+d)(c+d)(a+b)} \text{ when } ad - bc < 0$$

$$\chi^2 = \frac{N\left(ad - bc - \frac{1}{2}N\right)^2}{(a+c)(b+d)(c+d)(a+b)} \text{ when } ad - bc > 0.$$

ILLUSTRATIVE EXAMPLES

Example 1. *What are the expected frequencies of* 2×2 *contingency tables given below:*

(*i*)

a	*b*
c	*d*

(*ii*)

2	10
6	6

Sol.

(*i*) Observed frequencies

a	*b*	$a + b$
c	*d*	$c + d$
$a + c$	$b + d$	$a + b + c + d = N$

Expected frequencies

$\dfrac{(a+c)(a+b)}{a+b+c+d}$	$\dfrac{(b+d)(a+b)}{a+b+c+d}$
$\dfrac{(a+c)(c+d)}{a+b+c+d}$	$\dfrac{(b+d)(c+d)}{a+b+c+d}$

(*ii*) Observed frequencies

2	10	12
6	6	12
8	16	24

Expected frequencies

$\dfrac{8 \times 12}{24} = 4$	$\dfrac{16 \times 12}{24} = 8$
$\dfrac{8 \times 12}{24} = 4$	$\dfrac{16 \times 12}{24} = 8$

Example 2. *From the following table regarding the colour of eyes of fathers and sons test if the colour of son's eye is associated with that of the father.*

		Eye colour of son	
		Light	*Not light*
Eye colour of father	*Light*	*471*	*51*
	Not light	*148*	*230*

Sol. Null hypothesis H_0: The colour of son's eye is not associated with that of the father. *i.e.*, they are independent.

Under H_0, we calculate the expected frequency in each cell as

$$= \frac{\text{Product of column total and row total}}{\text{whole total}}$$

Expected frequencies are

Eye colour of father \ Eye colour of son	*Light*	*Not light*	*Total*
Light	$\frac{619 \times 522}{900} = 359.02$	$\frac{289 \times 522}{900} = 167.62$	522
Not light	$\frac{619 \times 378}{900} = 259.98$	$\frac{289 \times 378}{900} = 121.38$	378
Total	619	289	900

$$\chi^2 = \frac{(471-359.02)^2}{359.02} + \frac{(51-167.62)^2}{167.62} + \frac{(148-259.98)^2}{259.98} + \frac{(230-121.38)^2}{121.38} = 261.498.$$

Conclusion. Tabulated value of χ^2 at 5% level for 1 d.f. is 3.841.

Since the calculated value of $\chi^2 >$ tabulated value of χ^2, H_0 is rejected. They are dependent *i.e.*, the colour of son's eye is associated with that of the father.

Example 3. *The following table gives the number of good and bad parts produced by each of the three shifts in a factors*

	Goods parts	*Bad parts*	*Total*
Day shift	*960*	*40*	*1000*
Evening shift	*940*	*50*	*990*
Night shift	*950*	*45*	*995*
Total	*2850*	*135*	*2985*

Test whether or not the production of bad parts is independent of the shift on which they were produced.

Sol. Null hypothesis H_0. The production of bad parts is independent of the shift on which they were produced.

i.e., the two attributes, production and shifts are independent.

$$\text{Under } H_0, \qquad \chi^2 = \sum_{i=1}^{2} \sum_{j=1}^{3} \left[\frac{[(A_iB_j)_0 - (A_iB_j)]^2}{(A_i\,B_j)_0} \right]$$

Calculation of expected frequencies

Let A and B be the two attributes namely production and shifts. A is divided into two classes A_1, A_2. B is divided into three classes B_1, B_2, B_3.

$$(A_1B_1)_0 = \frac{(A_1)(B_2)}{N} = \frac{(2850)\times(1000)}{2985} = 954.77;$$

$$(A_1B_2)_0 = \frac{(A_1)(B_2)}{N} = \frac{(2850)\times(990)}{2985} = 945.226$$

$$(A_1B_3)_0 = \frac{(A_1)(B_3)}{N} = \frac{(2850)\times(995)}{2985} = 950;$$

$$(A_2B_1)_0 = \frac{(A_2)(B_1)}{N} = \frac{(135)\times(1000)}{2985} = 45.27$$

$$(A_2B_2)_0 = \frac{(A_2)(B_2)}{N} = \frac{(135)\times(990)}{2985} = 44.773;$$

$$(A_2B_3)_0 = \frac{(A_2)(B_3)}{N} = \frac{(135)\times(995)}{2985} = 45$$

To calculate the value of χ^2

Class	O_i	E_i	$(O_i - E_i)^2$	$(O_i - E_i)^2/Ei$
(A_1B_1)	960	954.77	27.3529	0.02864
(A_1B_2)	940	945.226	27.3110	0.02889
(A_1B_3)	950	950	0	0
(A_2B_1)	40	45.27	27.7729	0.61349
(A_2B_2)	50	44.773	27.3215	0.61022
(A_2B_3)	45	45	0	0
				1.28126

Conclusion. The tabulated value of χ^2 at 5% level of significance for 2 degrees of freedom $(r-1)(s-1)$ is 5.991. Since the calculated value of χ^2 is less than the tabulated value, we accept H_0. *i.e.*, the production of bad parts is independent of the shift on which they were produced.

Example 4. *From the following data, find whether hair colour and sex are associated.*

Sex \ *Colour*	*Fair*	*Red*	*Medium*	*Dark*	*Black*	*Total*
Boys	*592*	*849*	*504*	*119*	*36*	*2100*
Girls	*544*	*677*	*451*	*97*	*14*	*1783*
Total	*1136*	*1526*	*955*	*216*	*50*	*3883*

Sol. Null hypothesis. H_0: The two attributes hair colour and sex are not associated. *i.e.*, They are independent.

Let A and B be the attributes hair colour and sex respectively. A is divides into 5 classes ($r = 5$). B is divided into 2 classes ($s = 2$).

$\therefore$ Degrees of freedom $= (r-1)(s-1) = (5-1)(2-1) = 4$

Under H_0, we calculate $\chi^2 = \sum_{i=1}^{5} \sum_{j=1}^{2} \frac{[(A_iB_j)_0 - (A_iB_j)]^2}{(A_iB_j)_0}$

To calculate the expected frequency $(A_i B_j)_0$ as follows:

$$(A_1B_1)_0 = \frac{(A_1)(B_1)}{N} = \frac{1136 \times 2100}{3883} = 614.37;$$

$$(A_1B_2)_0 = \frac{(A_1)(B_2)}{N} = \frac{1136 \times 1783}{3883} = 521.629$$

$$(A_2B_1)_0 = \frac{(A_2)(B_1)}{N} = \frac{1526 \times 2100}{3883} = 852.289;$$

$$(A_2B_2)_0 = \frac{(A_2)(B_2)}{N} = \frac{1526 \times 1783}{3883} = 700.71$$

$$(A_3B_1)_0 = \frac{(A_3)(B_1)}{N} = \frac{955 \times 2100}{3883} = 516.482;$$

$$(A_3B_2)_0 = \frac{(A_3)(B_2)}{N} = \frac{955 \times 1783}{3883} = 483.517$$

$$(A_4B_1)_0 = \frac{(A_4)(B_1)}{N} = \frac{216 \times 2100}{3883} = 116.816;$$

$$(A_4B_2)_0 = \frac{(A_4)(B_2)}{N} = \frac{216 \times 1783}{3883} = 99.183$$

$$(A_5B_1)_0 = \frac{(A_5)(B_1)}{N} = \frac{50 \times 2100}{3883} = 27.04;$$

$$(A_5B_2)_0 = \frac{(A_5)(B_2)}{N} = \frac{50 \times 1783}{3883} = 22.959$$

Calculation of χ^2

Class	O_i	E_i	$(O_i - E_i)^2$	$\frac{(O_i - E_i)^2}{E_i}$
A_1B_1	592	614.37	500.416	0.8145
A_1B_2	544	521.629	500.462	0.959
A_2B_1	849	852.289	10.8175	0.0127
A_2B_2	677	700.71	562.1641	0.8023
A_3B_1	504	516.482	155.800	0.3016
A_3B_2	451	438.517	155.825	0.3553
A_4B_1	119	116.816	4.7698	0.0408
A_4B_2	97	99.183	4.7654	0.0480
A_5B_1	36	27.04	80.2816	2.9689
A_5B_2	14	22.959	80.2636	3.495
				9.79975

$\chi^2 = 9.799.$

Conclusion. Table of χ^2 at 5% level of significance for 4 d.f. is 9.488.

Since the calculated value of χ^2 < tabulated value H_0 is rejected. *i.e.*, the two attributes are not independent. *i.e.*, the hair colour and sex are associated.

Example 5. *Can vaccination be regarded as preventive measure of small pox as evidenced by the following data of 1482 persons exposed to small pox in a locality. 368 in all were attacked of these 1482 persons and 343 were vaccinated and of these only 35 were attacked.*

Sol. For the given data we form the contingency table. Let the two attributes be vaccination and exposed to small pox. Each attributes is divided into two classes.

Vaccination A / *Disease small pox B*	*Vaccinated*	*Not*	*Total*
Attacked	35	333	368
Not	308	806	1114
Total	343	1139	1482

Null hypothesis H_0. The two attributes are independent *i.e.*, vaccination can't be regarded as preventive measure of small pox.

Degrees of freedom $\nu = (r-1)(s-1) = (2-1)(2-1) = 1$

Under H_0, $\chi^2 = \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{[(A_iB_j)_0 - (A_iB_j)]^2}{(A_iB_j)_0}$

Calculation of expected frequency

$$(A_1B_1)_0 = \frac{(A_1)(B_1)}{N} = \frac{343 \times 368}{1482} = 85.1713;$$

$$(A_1B_2)_0 = \frac{(A_1)(B_2)}{N} = \frac{343 \times 1114}{1482} = 257.828$$

$$(A_2B_1)_0 = \frac{(A_2)(B_1)}{N} = \frac{1139 \times 368}{1482} = 282.828;$$

$$(A_2B_2)_0 = \frac{(A_2)(B_2)}{N} = \frac{1139 \times 1114}{1482} = 856.171$$

Calculation of χ^2

Class	O_i	E_i	$(O_i - E_i)^2$	$\frac{(O_i - E_i)^2}{E_i}$
(A_1B_1)	35	85.1713	2517.159	29.554
(A_1B_2)	308	257.828	2517.229	8.1728
(A_2B_1)	333	282.828	2517.2295	7.5592
(A_2B_2)	806	856.171	2517.1292	2.9399
				48.2261

Calculated value of $\chi^2 = 48.2261$.

Conclusion. Tabulated value of χ^2 at 5% level of significance for 1 d.f. is 3.841.

Since the calculated value of χ^2 > tabulated value H_0 is rejected.

i.e., the two attributes are not independent. *i.e.*, the vaccination can be regarded as preventive measure of small pox.

EXERCISE 6.9

1. In a locality 100 persons were randomly selected and asked about their educational achievements. The results are given below:

		Education		
		Middle	*High school*	*College*
Sex	Male	10	15	25
	Female	25	10	15

Based on this information can you say the education depends on sex.

2. The following data is collected on two characters:

	Smokers	*Non-smokers*
Literate	83	57
Illiterate	45	68

Based on this information can you say that there is no relation between habit of smoking and literacy.

3. 500 students at school were graded according to their intelligences and economic conditions of their homes. Examine whether there is any association between economic condition and intelligence, from the following data:

Economic conditions	*Intelligence*	
	Good	*Bad*
Rich	85	75
Poor	165	175

4. In an experiment on the immunisation of goats from anthrex, the following results were obtained. Derive your inferences on the efficiency of the vaccine.

	Died	*Survived*
Inoculated with vaccine	2	10
Not inoculated	6	6

Answers

1. Yes 2. No 3. No 4. Not effective.

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